Acoustics Crib Sheet

Linear Acoustics

Need three equations for lossless wave equation:

1. Continuity $\quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad (1)$
2. Momentum $\quad \rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + \nabla P = \mathbf{f} \qquad (2)$
3. Equation of State $\quad P = P(\rho, S) \qquad (3)$

Derive linear wave equation by letting $P = P_0 + p$, $\boldsymbol{v} = \boldsymbol{u}$, $\rho = \rho_0 + \rho'$, and retaining only first term of adiabatic (S constant) expansion of Eq. (3), so that $p = Ks$. Then discard products of acoustic quantities and using Eq. (1) to eliminate terms in Eq. (2). Solutions for planar and spherical harmonic ($\propto e^{-i\omega t}$, $k = \omega/c$) waves are

$$p(\boldsymbol{r}) = A\, e^{\pm i \boldsymbol{k}\cdot\boldsymbol{r}} \qquad \text{and} \qquad p(r) = A\frac{e^{\pm ikr}}{r}$$

The acoustic **impedance** $Z \equiv p/u$, where u is found from linearized momentum equation $\rho_0 \boldsymbol{u}_t = \nabla p$

$$Z = \pm\rho_0 c_0 \qquad Z = \rho_0 c_0 \left(1 - 1/ikr\right)^{-1} .$$

The small signal **speed of sound** is defined at adiabatic conditions $c^2 \equiv \left(\partial P/\partial \rho\right)_s = K/\rho$. For a perfect gas then $c_0 = \sqrt{\gamma R T} = \sqrt{\gamma P_0/\rho_0}$. More generally, we have PV^γ constant, and $[B/2A = (\gamma-1)/2]$

$$p = P_0\left[\gamma s + \frac{1}{2}\gamma(\gamma-1)s^2 + \dots\right]$$
$$\simeq \gamma P_0\, s + \frac{\gamma P_0}{2}(\gamma-1)s^2 = K\, s + \frac{B}{2A} K\, s^2 .$$

Acoustic Quanities

RMS quantities defined by $p_{\mathrm{rms}}^2 = (1/T)\int_0^T p^2(t)\,\mathrm{d}t$. SPL defined by $L_p \equiv 10\log_{10} p_{\mathrm{rms}}^2/p_{\mathrm{ref}}^2$. For incoherent sources, $p_{\mathrm{rms}}^2 = \sum_n p_{n,\mathrm{rms}}^2$. If phase difference ϕ between 2 sources, $p_{\mathrm{rms}}^2 = p_{1,\mathrm{rms}}^2 + p_{2,\mathrm{rms}}^2 + P_1 P_2 \cos\phi$. For harmonic sources, $p_{\mathrm{rms}} = P_0/\sqrt{2}$.

1. Energy Density [J/m^3] $\quad \mathcal{E} = \frac{1}{2}\rho_0\left[u^2 + (p/\rho_0 c_0)^2\right]$
2. Intensity [W/m^3] $\quad \boldsymbol{I} = p\boldsymbol{u}$
   (a) Progressive waves: $\quad \langle I\rangle = P^2/2\rho_0 c_0$
   (b) Harmonic Waves: $\quad \langle I\rangle = \frac{1}{2}\operatorname{Re} PU^*$
3. Power [W] $\quad W = \int_S \boldsymbol{I}\cdot\boldsymbol{n}\,\mathrm{d}S$
4. Directivity [Unitless] $\quad D = D(\theta, \phi)$

Reflection and Transmission

In all cases, we need continuity of normal particle velocity. Usually we also have continuity of pressure (or there is a pressure differential). Acoustic impedance for plane waves is $Z_{\mathrm{ac}} \equiv \rho_0 c_0/\cos\theta = \rho_0 c_0/S$, and it turns out that the reflection coefficient is

$$\mathcal{R} = (Z_{\mathrm{ac},2} - Z_{\mathrm{ac},1})/(Z_{\mathrm{ac},2} + Z_{\mathrm{ac},1}) .$$

If the second medium is harder, we have a **critical angle**

$$\sin\theta_c = c_1/c_2 , \qquad \cos\theta_t = \sqrt{1 - (c_2/c_1)\sin^2\theta_i} .$$

Power reflection and transmission coefficients are governed by

$$r = \|\mathcal{R}\|^2 \qquad \tau + r = 1 .$$

The **angle of intromission** is when $r = 0$

$$\sin\theta_p = \left[1 - (Z_1/Z_2)^2\right] \Big/ \left[1 - (\rho_1/\rho_2)^2\right] .$$

For a locally reacting surface $Z_{\mathrm{ac},2} = Z_n = r_n - ix_n$ so that

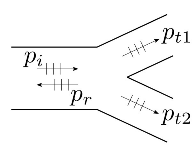
$$\mathcal{R} = (Z_n - Z_{\mathrm{ac},2})/(Z_n + Z_{\mathrm{ac},2}) .$$

The impedance seen looking into a termination is

$$Z_{\mathrm{in}} = iZ_0 \cot kL \text{ (rigid)} \qquad Z_{\mathrm{in}} = -iZ_0 \tan kL \text{ (free)}$$

If we have a pressure discontinuity, then write a force balance

$$\begin{aligned} p_i + p_r - p_t &= (m/S)\ddot{\xi} = (m/S)\dot{u} && \text{(mass law)}\\ &= r_f u && \text{(flow resistance)}\\ &= m\dot{u}\left[1 - (v\sin\theta_i/c_0)^4\right] && \text{(stiff plate).} \end{aligned}$$



If there is a change in area, need continuity of pressure and total volume velocity:

$$p_i + p_r = p_{t1} = p_{t2}$$
$$S_1(u_i + u_r) = S_1 u_{t1} + S_2 u_{t2}$$

In this case we'd find

$$\mathcal{R} = (Z_{\mathrm{ac},0}^{-1} - Z_{\mathrm{ac},1}^{-1} - Z_{\mathrm{ac},2}^{-1})/(Z_{\mathrm{ac},0}^{-1} + Z_{\mathrm{ac},1}^{-1} + Z_{\mathrm{ac},2}^{-1}) .$$

Waveguides

For a rectangular (z infinite) waveguide,

$$p = \sum_{n,m} C_{mn} \left\{ \begin{array}{c} \cos k_x x \\ \sin k_x x \end{array} \right\} \left\{ \begin{array}{c} \cos k_y y \\ \sin k_y y \end{array} \right\} e^{i\sqrt{\omega^2/c_0^2 - k_x^2 - k_y^2}\, z}$$

and for a rigid circular duct with radius a ($k_r = \alpha'_{mn}/a$)

$$p = \sum_{n,m} C_{mn} J_n(k_r r) \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} e^{i\sqrt{\omega^2/c_0^2 - k_r^2}\, z}$$

In both cases the phase speed is $c_p = \omega/k_z$ and the speed at which energy propagates is $\mathrm{d}\omega/\mathrm{d}k_z$. The **cutoff frequency** for a mode is ω such that the propagating wavenumber is real.

Radiation

The Rayleigh Integral is ($\mathcal{r} = \|\boldsymbol{r} - \boldsymbol{r}'\|$)

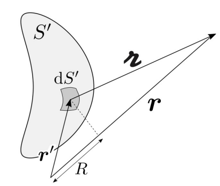
$$p(\boldsymbol{r}) = -\frac{i\omega\rho_0}{4\pi}\int_{S'} \frac{e^{ik\mathcal{r}}}{\mathcal{r}} \boldsymbol{u}(\boldsymbol{r}')\cdot\boldsymbol{n}\,\mathrm{d}S' . \qquad (4)$$

Now define $\boldsymbol{e}_r = \boldsymbol{r}/\|\boldsymbol{r}\|$, and $R = \boldsymbol{r}'\cdot\boldsymbol{e}_r$, then for phase

$$\begin{aligned} \mathcal{r} &= \left[(\boldsymbol{r} - \boldsymbol{r}')\cdot(\boldsymbol{r} - \boldsymbol{r}')\right]^{\frac{1}{2}} = r\left[1 - (r'/r)^2 + 2\,(\boldsymbol{r}\cdot\boldsymbol{r}')/r^2\right]^{\frac{1}{2}} \\ &\simeq r\left[1 - R/r + \mathcal{O}\left(r'^2/r^2\right)\right] = r - R , \qquad (5) \end{aligned}$$

and $\mathcal{r} \simeq r$ for the denominator. Eq. (5) $\rightarrow$ Eq. (4) gives

$$\begin{aligned} p(\boldsymbol{r}) &\simeq -i\omega\rho_0 \frac{e^{ikr}}{4\pi r}\int_{S'} e^{-ikR}\boldsymbol{u}(\boldsymbol{r}')\cdot\boldsymbol{n}\,\mathrm{d}S' \\ &\simeq -i\omega\rho_0 \frac{e^{ikr}}{4\pi r}\int_{S'} \left[1 + ikR + \frac{1}{2}(ikR)^2\right]\boldsymbol{u}(\boldsymbol{r}')\cdot\boldsymbol{n}\,\mathrm{d}S' . \qquad (6) \end{aligned}$$



The first, second, and third terms of Eq. (6) are the monopole, dipole and quadrupole terms. Note that for a monopole source, the dipole term cancels since there is a $-R$ point for every R. For a dipole, the monopole term vanishes, since there is no net fluid flow through S'.

For a **circular piston** of radius a, the on-axis and far-field evaluation of Eq. (4) give (respectively)

$$p(z, t) = \rho_0 c_0 u_0 \left(e^{ikz} - e^{ik\sqrt{z^2 + a^2}}\right) e^{-i\omega t} ,$$
$$p(r, \theta, t) = -\frac{i\rho_0 c_0 u_0 R}{\pi r \sin\theta} J_1(ka\sin\theta)\, e^{-i(\omega t - kr)} .$$

The radiation impedance seen by the piston is

$$Z_{\mathrm{rad}} = \rho_0 c_0 \left[R_1(2ka) - iX_1(2ka)\right] .$$

In the far field, acoustic monopole and dipole field amplitudes are,

$$|p(r,\theta)| = A \quad \text{and} \quad |p(r,\theta)| = Akd\cos\theta$$

where $A = -i\omega\rho_0 c_0/4\pi r$. The lateral and longitudinal quadrupole fields are

$$|p(r,\theta)| = A(kd)^2\cos\theta\sin\theta \quad \text{and} \quad |p(r,\theta)| = A(kd)^2\cos^2\theta .$$