Math 6701 Exam 3 Review

Series Solutions

Power Series

Provided x_0 is not a singular point, a continuously differential function may be expanded about x_0 as

$$\begin{aligned} y(x) &\simeq y(x_0) + y'(x_0)(x-x_0) + \frac{1}{2}y''(x_0)(x-x_0)^2 + \dots \\ &\simeq \sum_{n=0}^{\infty} a_n (x-x_0)^n . \end{aligned} \tag{1}$$

A series is **convergent** on the region where

$$|x-x_0| \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1 . \tag{2}$$

Thus, if we have some differential equation

$$y'' + p(x)y' + q(x)y = 0 , \tag{3}$$

we can find two linearly independent power series solutions by substituting Eq. (1) into Eq. (3). To combine series, get the powers of $(x-x_0)$ to match (n is just a dummy index), e.g.,

$$\begin{aligned} &\sum_{n=3}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} \quad \Big| \text{ let } \begin{matrix} j = n-2 \\ k = n+1 \end{matrix} \\ &= \sum_{j=1}^{\infty} (j+2)(j+1)a_{j+2}x^j + \sum_{k=1}^{\infty} c_{k-1}x^k \\ &= \sum_{\ell=1}^{\infty} \Big[(\ell+2)(\ell+1)a_{\ell+2} + c_{\ell-1}\Big] x^\ell . \end{aligned}$$

This will give a recurrence relationship with either two or three terms. If three terms, e.g.,

$$c_{n+2} = f(n)\, c_n + g(n)\, c_{n+1}$$

set $c_n = 1$ and $c_{n+1} = 0$, then $c_n = 0$ and $c_{n+1} = 1$. This will allow us to solve for any

Frobenius Method

We may solve Eq. (3) about a singular point x_0 (note that the coefficient of y'' is unity!) provided x_0 is a **regular singular point**. A singular point is called regular if

$$P(x) = (x-x_0)p(x) \quad \text{and} \quad Q(x) = (x-x_0)^2 q(x)$$

are analytic at x_0. If they are, then Eq. (3) has at least one solution of the form

$$y = (x-x_0)^r \sum_{n=0}^{\infty} a_n (x-x_0)^n . \tag{4}$$

The **indicial equation** has two roots

$$r(r-1) + a_0 r + b_0 = 0 , \tag{5}$$

where a_0 and b_0 are the first terms of the series expansion of $P(x)$ and $Q(x)$, respectively. If the two roots differ by an integer, a second solution may be given by $y_2 = y_1(x)\ln x$. If the two roots are the same, this form is guaranteed. Otherwise, there will be two independent solutions of the form of Eq. (4).

Complex Analysis

A complex number $z \in \mathbb{C}$ has a real and imaginary part

$$z = x + iy , \tag{6}$$

where $x, y \in \mathbb{R}$. In polar form

$$z = re^{i\theta} , \quad \text{where} \quad r = \sqrt{x^2+y^2} , \quad \text{and} \quad \theta = \arcsin\frac{y}{x} .$$

Here, $r = |z|$ is called the **modulus** of z and θ is its **argument**. Since θ is periodic, we call the value of $\theta \in (-\pi, \pi]$ the *principal argument* of z, denoted $\operatorname{Arg} z$.

DeMoivre's theorem implies that

$$z^n = r^n (\cos n\theta + i \sin n\theta) .$$

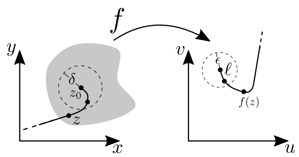
By writing $w^n = z$, solving for w gives the n^{th} roots of z [where $k \in \{\mathbb{Z} \cap (0, n-1)\}$]

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right] \tag{7}$$

Functions of a Complex Variable

A function f may map a complex number to another complex number $f : \mathbb{C} \to \mathbb{C}$. Like numbers, they may be written in terms of their real and imaginary parts

$$f(z) = u(x,y) + iv(x,y) . \tag{8}$$



If f is defined in a neighborhood of a point z_0, then the **limit** ℓ of the function exists if and only if for all $\epsilon > 0$, there exists some δ such that $|f(z) - \ell| < \epsilon$ when $|z - z_0| < \delta$.

The function is **continuous** at z_0 if $\ell = f(z_0)$. If the **derivative** of a complex valued function is defined

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} \tag{9}$$

exists, then f is said to be differentiable at that point. If $f(z)$ is differentiable at z, then the first partial derivatives u and v exist and the **Cauchy-Riemann equations** are satisfied:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} , \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} , \quad \text{or} \tag{10}$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} , \quad \text{and} \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} . \tag{11}$$

Complex functions are called **analytic** if they are continuous and differentiable in some region. Functions which are everywhere analytic are called **entire**. The only bounded, entire functions are constants.

The logarithm is extended to the complex plane by defining it as the inverse of exponentiation: $e^z = w \to \ln z = w$. Then

$$\ln z = \ln|z| + i(\theta + 2\pi n) . \tag{12}$$

The principal value of Eq. (12) is denoted

$$\operatorname{Ln} z = \ln|z| + i \operatorname{Arg} z ,$$

where $\operatorname{Arg} z \in (-\pi, \pi]$. Note that $\operatorname{Arg} z$ is *not analytic* on any region including the negative real axis, since there is a discontinuity as the axis is crossed. Real logarithm properties do not necessarily hold, e.g., $\operatorname{Ln} z^n \neq n \operatorname{Ln} z$ in general. To check, write in polar form, and note that

$$\operatorname{Ln} re^{i\theta} = \ln r + i\theta .$$

Contour Integration

A contour integral may be evaluated as a path integral in the complex plane

$$\int_C f(z)\,\mathrm{d}z = \int_{t_1}^{t_2} f[z(t)]\, z'(t)\,\mathrm{d}t . \tag{13}$$

In the complex plane, a circle or radius R centered at z_0 can be parameterized as $z(t) = z_0 + Re^{it}$, with $t \in [0, 2\pi]$. If f is analytic in R, then $f(z)$ has an antiderivative F and Eq. (13) is independent of the path C *and its parameterization*

$$\int_C f(z)\,\mathrm{d}z = F(z_1) - F(z_0) \quad (C \in R) . \tag{14}$$

The **Cauchy-Goursat theorem** says that if $f(z)$ is analytic and in a simply connected domain R, then for all $C \in R$

$$\oint_C f(z)\,\mathrm{d}z = 0 . \tag{15}$$

Consider for $a, b \in \mathbb{C}$, with m and n as positive integers,

$$f(z) = \frac{(z-a)^n}{(z-b)^m} . \tag{16}$$

The point $z = a$ is said to be a **zero** of order n, and $z = b$ a **pole** of order m. Now if f is analytic in a simply connected domain R, and $C \in R$ is a simple *closed* contour, then

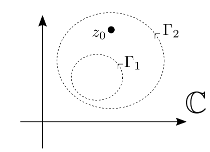
$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,\mathrm{d}z \quad \text{and} \tag{17}$$

$$f^{(n-1)}(z_0) = \frac{(n-1)!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^n}\,\mathrm{d}z , \tag{18}$$

where z_0 is a point on the interior of C. For instance, in the figure to the right, if f is analytic in R ($\partial R = \Gamma_2$), then

$$\oint_{\Gamma_1} \frac{f(z)}{z-z_0}\,\mathrm{d}z = 0 , \quad \text{while}$$

$$\oint_{\Gamma_2} \frac{f(z)}{z-z_0}\,\mathrm{d}z = 2\pi i f(z_0) .$$



For instance, to integrate $f(z) = z^2/(z-i)$ around a circle at the origin with radius 2, we set compute

$$\oint_C \frac{z^2}{z-i}\,\mathrm{d}z = 2\pi i \left(z^2\Big|_{z=i}\right) = 2\pi\left(i^2\right) = -2\pi i .$$

If $f(z)$ has multiple poles in R, then use partial fractions and integrate each term separately with Eqs. (17) and (18)

$$\frac{az^2+bz+c}{(z-z_1)(z-z_2)^2} = \frac{A}{z-z_1} + \frac{B}{z-z_2} + \frac{C}{(z-z_2)^2} .$$