Math 6646 Exam 1 Review

Definitions

A function f continuous on an open connected set $\mathcal{D} \in \mathbb{R}^2$ satisfies the **Lipschitz condition** if

$$|f(t, x_1) - f(t, x_2)| \leq K|x_1 - x_2|$$

for all (t, x_1), $(t, x_1) \in \mathcal{D}$. **NB** If f is Lipschitz continuous, then there exists a unique solution to $x' = f$ with $x(t_0) = x_0$ in a finite interval including t_0.

A value z is said to be of **order** p [$z = \mathcal{O}(h^p)$] if there exist positive constants h_0 and C such that $|z| \leq Ch^p$ for all $h \in (0, h_0)$.

A scheme is said to be **convergent** on an interval $\mathcal{I}$ if, for all $t_n \in \mathcal{I}$, $|x_n - x(t_n)| \to 0$ as $h \to 0$.

A scheme is said to be **consistent** if its difference operator $\mathcal{L}$ has finite positive order p. Consistency implies convergence.

A linear multistep method is **zero-stable** if all roots of its characteristic polynomial $\rho(r)$ are such that $|r| \leq 1$, and any $r = 1$ are simple roots. A method is zero-stable if and only if it is consistent and convergent.

An LMM is **absolutely stable** if its application to $x' = \lambda x$ (with $\operatorname{Re}\lambda < 0$) with a given value of $\hat{h} = h\lambda$, its solutions tend to 0 as $n \to 0$ for any starting values.

An LMM is **A-stable** if its region of absolute stability includes the entire left half plane (i.e., for all $\operatorname{Re}\hat{h} < 0$).

Taylor Series Approximations

Only first-order ODEs will be discussed, since any higher order linear ODE may be written as a system of these first order problems. Consider

$$x''' + ax'' + bx' + cx = f.$$

Now if we let $y = x'$ and $z = y' = x''$, then we can write

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -(az + by + cx) + f \end{aligned}$$

Euler's Method

Euler's method solves the IVP $x'(t) = f(x, t)$, $x_0 = \eta$ with the scheme

$$x_{n+1} = x_n + hf_n\,. \tag{1}$$

Since $x(t+h) = x(t) + hx'(t) + \frac{1}{2!}h^2x''(t) + \mathcal{O}(h^3)$, we see that remainder terms have order $\mathcal{O}(h^2)$.

Proof of Convergence Consider the IVP for some constant λ and function $g(t)$

$$x'(t) = \lambda x + g\,, \quad x_0 = 1\,.$$

Euler's method gives

$$x_{n+1} = \lambda x_n + g(t_n)\,.$$

A Taylor series expansion of the solution gives $x(t_{n+1}) = x(t_n) + hx'(t_n) + \mathcal{O}(h^2)$, so defining the error $e_{n+1} \equiv x(t_n) - x_n$ gives

$$e_{n+1} = (1 + h\lambda)e_n + T_{n+1}\,,$$

where $T = \mathcal{O}(h^2)$ is error due top the truncation of the Taylor series. Since we know the initial condition, $e_0 = 0$, and then the error at each subsequent point is

$$\begin{aligned} e_{n+1} &= (1 + h\lambda)e_n + T_{n+1} \\ \implies e_n &= \sum_{j=1}^{n} (1 + h\lambda)^{n-j} T_j\,, \end{aligned}$$

Then, since $|1 + h\lambda| \leq 1 + h\lambda \leq e^{h|\lambda|}$ [note from the expansion that $e^x = 1 + x + \mathcal{O}(x^2)$],

$$|1 + h\lambda|^{n-j} \leq e^{h|\lambda|(n-j)} = e^{|\lambda|t_{n-j}} \leq e^{|\lambda|t_f}\,,$$

where $t_f = nh$ is the final time. Then, since by definition $|T_j| \leq Ch^2$ for some finite C, we can say that each term of Eq. (2) is bounded by $(Ce^{|\lambda|t_f})h^2$, and thus

$$|e_n| \leq n\left(Ce^{|\lambda|t_f}h^2\right) = nh\,hCe^{|\lambda|t_f} = t_f\,hCe^{|\lambda|t_f}$$

Higher Order Methods

Higher-order accuracy can be achieved by retaining more terms in the Taylor series, i.e.,

$$x_{n+1} = x_n + hf_n + \frac{h^2}{2!}f_n' + \frac{h^3}{3!}f_n'' + \ldots\,. \tag{2}$$

This assumes that f_n is continually differentiable, and such derivatives may not be easy to determine.

Linear Multistep Methods

To avoid having to determine analytical derivatives (as second and higher-order Taylor series methods require), multi-step methods approximate these derivatives with known values. Consider that for any function $z(t)$ whose first three derivatives are defined, we can write

$$z'(t+h) = z'(t) + hz''(t) + \mathcal{O}(h^2)\,. \tag{3}$$

Then expanding $z(t)$ and using Eq. (3),

$$\begin{aligned} z(t+h) &= z(t) + hz'(t) + \frac{h^2}{2}z''(t) + \mathcal{O}(h^3) && (4) \\ &= z(t) + hz'(t) + \frac{h^2}{2}\left\{\frac{1}{h}\left[z'(t+h) - z'(t)\right]\right\} + \mathcal{O}(h^3) \\ &= z(t) + \frac{h}{2}\left[z'(t+h) + z'(t)\right] + \mathcal{O}(h^3)\,. && (5) \end{aligned}$$

If we have an ODE $x' = f$ we're trying to solve, then the scheme gives the *trapezoidal rule*

$$x_{n+1} = x_n + \frac{h}{2}\Big(f_{n+1} + f_n\Big)\,. \tag{6}$$

Now consider the expansion for $z'(t-h) = z'(t) - hz''(t) + \mathcal{O}(h^2)$, so that re-arranging and substituting into Eq. (4) gives (omitting the algebra)

$$z(t+h) = z(t) + \frac{h}{2}\left[3z'(t) - z'(t-h)\right] + \mathcal{O}(h^3)\,, \tag{7}$$

or as a scheme

$$x_{n+1} = x_n + \frac{h}{2}\Big(3f_n + f_{n-1}\Big)\,. \tag{8}$$

This scheme requires values at t_{n-1} *and* t_n to compute the value at t_{n+1} and is thus a *multistep method*. However, they have the benefit of having second-order accuracy.

Functional Iteration

Implicit methods (i.e., those which require knowledge of f_{n+1}) will yield a nonlinear equation whose roots we need to know. One way to find these is *functional iteration*, where we start with an initial guess, (say $x_{n+1}^{[0]} \approx x_n$), and then plug the result into the initial expression. So for backward Euler

$$x_{n+1}^{[j+1]} = x_n + hf\left(t_{n+1}, x_{n+1}^{[j]}\right) \tag{9}$$

Consistency, Convergence, and Zero-Stability

Two step LMMs can be written most generally as

$$x_{n+2} + \alpha_1 c_{n+1} + \alpha_0 x_n = h\Big(\beta_2 f_{n+2} + \beta_1 f_{n+1} + \beta_0 f_n\Big)\,.$$

A scheme is *implicit* if $\beta_2 \neq 0$ and *explicit* otherwise. The **difference operator** $\mathcal{L}$ for the scheme is

$$\begin{aligned} \mathcal{L} \equiv\ & z(t+2h) + \alpha_1 z(t+h) + \alpha_0 z(t) \\ & - h\Big[\beta_2 z'(t+2h) + \beta_1 z'(t+h) + \beta_0 z'(t)\Big]\,. \end{aligned} \tag{10}$$

By Taylor expanding each term, we can find the order p of $\mathcal{L} = \mathcal{O}(h^{p+1})$. If $p > 0$, then the method is **consistent**. To test for consistency, assemble the characteristic polynomials

$$\rho(r) = r^2 + \alpha_1 r + \alpha_0 \quad \text{and} \quad \sigma(r) = \beta_2 r^2 + \beta_1 r + \beta_0\,. \tag{11}$$

The method is consistent if and only if $\rho(1) = 1$ and $\rho'(1) = \sigma(1)$. In k-step cases, this condition becomes

$$\sum_{j=0}^{k} \alpha_j = 0 \quad \text{and} \quad \sum_{j=0}^{k} j\alpha_j = \sum_{j=0}^{k} \beta_j$$

Methods whose characteristic polynomial [$\rho(r)$ in Eq. (11)] has roots with magnitude less than 1 (root condition) are **zero-stable**. If a method is consistent is consistent and zero-stable, then it is convergent. **Absolute stability** is is true when $p(r) = \rho(r) - h\lambda\sigma(r)$ obeys the root condition.