ME 8873 - Georgia Tech

Lecture 11: Focused Ultrasonic Fields

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Preliminaries

A few preliminaries that will be invoked along the way

1. **Wave equation is linear**
   Thus if we have two sources that radiate fields p_1 and p_2, the total field is just $p = p_1 + p_2$.

   What happens if first source strength is doubled?
   Then simply $p = p_1 + p_1 + p_2 = 2p_1 + p_2$. Termed *Superposition*.

2. **All signals are time harmonic**
   This is not really a restriction, since an arbitrary signal can be made up of appropriately weighted sinusoids. Since we can write $\cos \omega t = \operatorname{Re} e^{-i\omega t}$, we will take the pressure as the real part of the complex quantity.

   (a) Negative time convention is used $p(\boldsymbol{r}, \omega) = p(\boldsymbol{r})e^{-i\omega t}$.
   (b) Time derivatives become multiplications: $\partial/\partial t \to -i\omega$

   Pressure can be imaginary?
   No, the pressure will be the real part of the answer. Can't measure imaginary quantities.

   Consequences for 1D wave equation?
   W.E. becomes $\partial^2 p/\partial x^2 + k^2 x = 0$, where $k \equiv \omega/c$. Called *Helmholtz equation*.

3. **Velocity Potential is Convenient**

   (a) By definition $\boldsymbol{u} = \nabla\phi$
   (b) As a consequence, $p = \rho_0 \frac{\partial \phi}{\partial t} = -i\omega\rho_0\phi$.

   Does ϕ obey the wave equation?
   Yes! Proportional to pressure, and wave equation is linear.

4. **There are no boundaries**
   This obviously is a restriction, but a reasonable one as long as the closest boundary at $r = \ell$ is such that $\ell/d \gg 1$.

1 Modeling Complex Sources

1.1 Point Sources

We saw from homework 1 that a single outgoing point source (no boundaries) can described by

$$p(\boldsymbol{r},t) = p(r,t) = \frac{p(t - r/c)}{r} \,. \tag{1}$$

Since our source is periodic, we have

$$\begin{aligned} p(r,t) &= p_0 \, \frac{e^{-i\omega(t-r/c)}}{r} \\ &= p_0 \, \frac{e^{i(\omega/c)r}}{r} \, e^{-i\omega t} \,. \end{aligned} \tag{2}$$

The quantity $\omega/c \equiv k$ is called the ***wavenumber***. Since it's a pain to keep having to write $e^{-i\omega t}$ all the time, let's just take it as implied. Thus the field due to the point source is

$$p(r) = p_0 \, \frac{e^{ikr}}{r} \,. \tag{3}$$

1.2 Extended Sources

Okay, so this Eq. (3) gives the field for a point source. But what if we have something more complicated? Huygen's-Fresnel principle says that "every point on a wavefront is itself the source of spherical wavelets, and the secondary wavelengths emanating from different points mutually interfere. The sum of these spherical wavelets forms the wavefront." While merely a postulate, the principle successfully predicts effects such as refraction and diffracation (see Fig. 1).

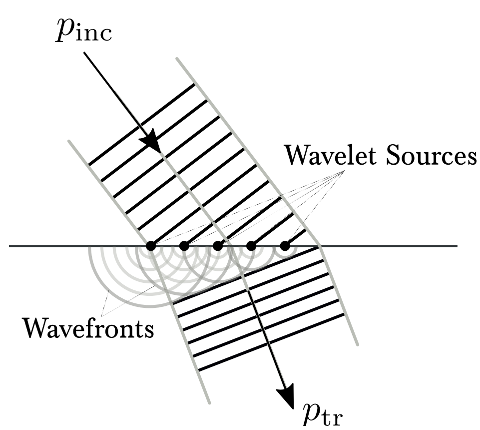


Figure 1: Huygen's-Fresnel principle predicts refraction

So if we want to model a more complicated source, we could perhaps just approximate it as a bunch of point sources. Then, since the wave equation is linear, we can just add up the field due to all these point sources!

1.3 The Rayleigh Integral

Though it can be shown rigorously (see Sec. 4.1), the idea of computing the field due to a continuous distribution of point sources is bourne out by the Rayleigh integral

$$p(\boldsymbol{r}) = \frac{-i\omega\rho_0}{2\pi} \int_S u(\boldsymbol{r}') \frac{e^{ikR}}{R} \, \mathrm{d}R \tag{4}$$

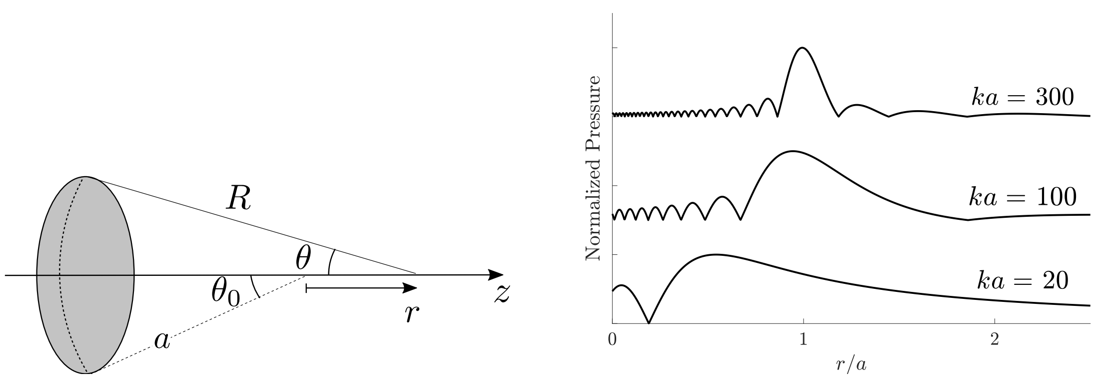
where S is the surface, $\boldsymbol{r}'$ is the position vector to the surface, and $R = |\boldsymbol{r} - \boldsymbol{r}'|$. Basically, we're attempting to sum the contribution of infinitesimal point sources over the surface of the radiator.

The difficulty in evaluating Eq. (4) which shows up a lot in mathematical physics is due to the presence of R, and the difficulty in choosing an appropriate geometry. Usually we have to make an approximation (e.g., far field) or choose a convenient geometry (e.g., along the axis). Or both.

2 Special Cases

It turns out that the convenient integral is over the relative distance magnitude R. The integration is a nontrivial geometry exercise (see Sec. 4.3), so we'll consider just some convenient cases.

2.1 On-Axis Field



(a) **(b)**

Figure 2: **(a)** Geometry for the on-axis acoustic field. **(b)** Normalized on-axis pressure for the indicated value of ka, normalized by the radius of curvature a.

If we restrict to on-axis field the the integartion becomes

$$p(r) = -i\omega\rho_0 u_0 \left(\frac{a}{r}\right) \int_{R_{\min}}^{R_{\max}} e^{ikR}\,\mathrm{d}R\,. \tag{5}$$

We'll need to integrate from

$$R_{\min} = a - r \tag{6}$$

$$R_{\max} = \sqrt{a^2 + r^2 - 2ar\cos\theta_0} \tag{7}$$

Thus the pressure takes the form

$$p(r) = -\rho_0 c_0 u_0 \left(\frac{a}{r}\right) \left[e^{ikR_{\max}} - e^{ikR_{\min}}\right]. \tag{8}$$

Example 1: Axial Field Characteristics

What do we notice from Eq. (8)?
Ans: Pressure proportional to radius of curvature and velocity; inversely proportional to distance.
Looks like two plane waves (direct and edge, similar to piston).

How to Calculate the Velocity?
Perhaps the easiest way to find the velocity is to note that by definition

$$p = \rho_0 \frac{\partial\phi}{\partial t} = -i\omega\rho_0\phi\,, \tag{9}$$

so by inspection of Eq. (8), we can see that

$$\phi = \frac{u_0 c_0}{i\omega}\frac{a}{r}\left[e^{ikR_{\max}} - e^{ikR_{\min}}\right], \tag{10}$$

and the velocity will be given by

$$\boldsymbol{u} = \nabla\phi. \tag{11}$$

The **gain** of the transducer is defined by

$$G \equiv \frac{|p_{\text{focal}}|}{\rho_0 c_0 u_0} = ka(1 - \cos\theta_0) \tag{12}$$

Typical gains are on the order of 10–30. It turns out that the pressure peak is not at the geometric focus

2.2 Radial Dependence

In the focal plane (i.e., for $z = 0$), it can be shown that the pressure obeys

$$p(r) = 2p(0)\,\frac{J_1(2kr\cos\theta_0)}{2kr\cos\theta_0}\,. \tag{13}$$

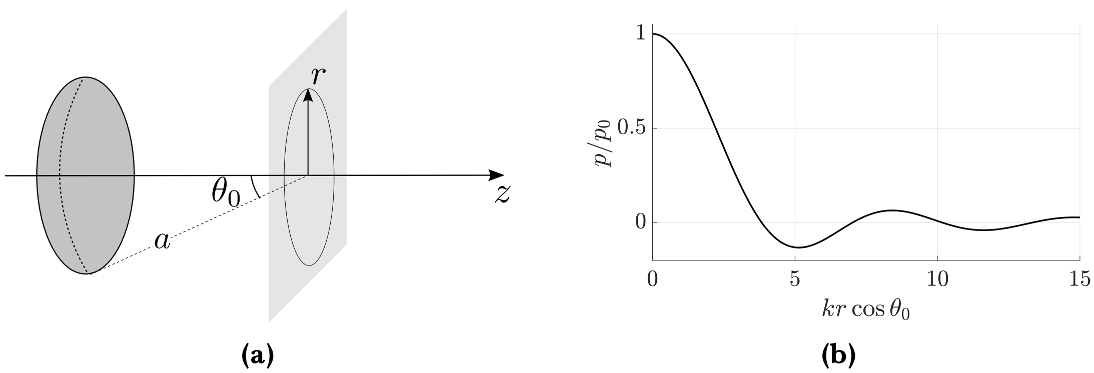


Figure 3: (a) Geometry for transverse field. **(b)** Normalized pressure field as a function of the normalized distance from the axis.

Example 2: Focal Plane Characteristics

What do we notice from Eq. (13)?
Ans: Scaling $(2k\cos\theta_0)$ of argument $\rightarrow$ tighter focus for higher frequency, smaller angles.

Why not always use high frequency for better focusing?
Ans: Attenuation. Since $\alpha \propto f^2$, higher frequencies quickly incur losses.

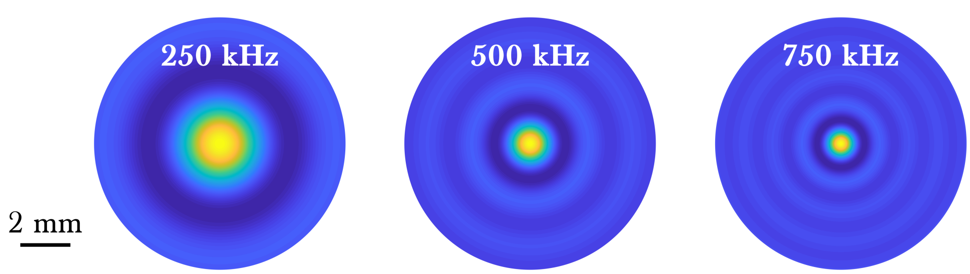
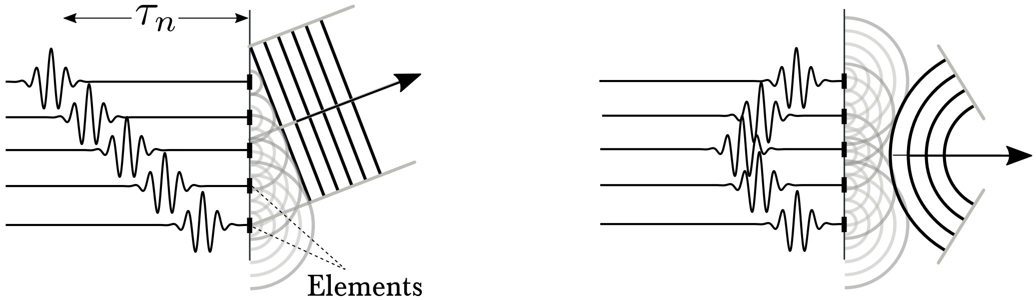


Figure 4: Normalized focal fields for the given frequency with $a = 40\,\mathrm{mm}$ and $\theta_0 = 30^\circ$.

3 Phased Arrays

A spherical transducer ensures that the sound leaves the surface such that the distance to the focal spot is the same for any part of the surface. This means if we want to change the focal position, we have to physically move the transducer.



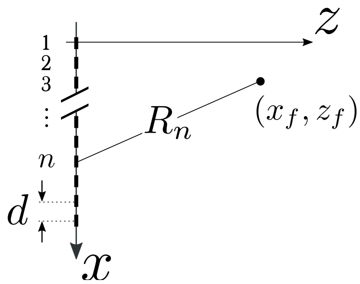
(a) **(b)**

Figure 5: **(a)** Steering and **(b)** focusing with a phased array.

However, suppose we have an array of individual elements we can control individually. If we are able to control the time delay of each element, then we can steer the emitted radiation (Fig. 5a) or achieve focusing (Fig. 5b). A huge advantage of phased arrays is that we can focus to arbitrary locations, not necessarily just along the axis.

Example 3: Focusing at Arbitrary Point

What are the time delays to focus at are arbitrary point (x_f, z_f)**?**
Ans: Well, we want all the waves to arrive in phase (i.e., at the same time). So the delays will have to be equal to the travel time from each transmitter to the desired focus. Call the plane of the transducer the x-axis, and the perpendicular direction the z.



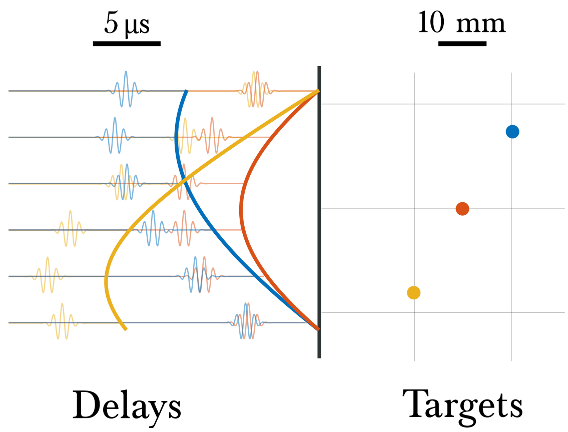
The distance to the focal point for the n^{th} transmitter is

$$R_n = \sqrt{[x_f - (n-1)d]^2 + z_f^2}\,.$$

Then, the n^{th} delay will be given by

$$\tau_n = R_n / c_0\,.$$

Example 3 (*cont.*)



4 Details

4.1 Obtaining the Rayleigh Integral

Consider the vector identity[1]

$$\nabla \cdot (f\nabla g - g\nabla f) = f\left(\nabla^2 + a^2\right) g - g\left(\nabla^2 + a^2\right) f\,, \tag{14}$$

where f and g are arbitrary functions of space and a is a constant (which may be 0, in which case this identity is more immediately evident). Now for some clever manipulations.

For reasons that will become clearer, let f be the Green's function $G = \exp(ikR)/R$, let g be the harmonic pressure field p, and let our arbitrary constant be the wavenumber $k = \omega/c_0$. And suppose all of our (harmonic, but otherwise arbitrary) sources are confined within some region bounded by a surface S. Then, let's integrate both sides over the volume contained between S and a large sphere of radius D (see Fig. 6).

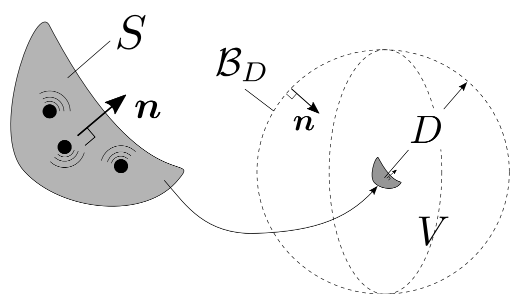


Figure 6: The surface S bounds all sources, while V is the region exterior to S and within a large sphere of radius D.

$$\underbrace{\int_V \nabla\cdot(G\nabla p - p\nabla G)\,\mathrm{d}V}_{\mathrm{I}} = \underbrace{\int_V G\left(\nabla^2 + k^2\right) p\,\mathrm{d}V}_{\mathrm{II}} - \underbrace{\int_V p\left(\nabla^2 + k^2\right) G\,\mathrm{d}V}_{\mathrm{III}}\,. \tag{15}$$

Immediately, we can say that term II of Eq. (15) vanishes, since there are no sources in V, and thus $(\nabla^2 + k^2)p = 0$. For the term III, we'll use the fact that $(\nabla^2 + k^2)G = -4\pi\delta(\boldsymbol{r})$ by definition, so

$$\begin{aligned}\int_V G\left(\nabla^2 + k^2\right) p\,\mathrm{d}V &= 0 - \int_V p\,\left[-4\pi\delta(\boldsymbol{r})\right]\,\mathrm{d}V \\ &= -4\pi p(\boldsymbol{r})\,,\end{aligned} \tag{16}$$

where $\boldsymbol{r}'$ lies on the surface S. Finally, we'll use the divergence theorem to write the term I of Eq. (15) as

$$\begin{aligned}\int_V \nabla\cdot(G\nabla p - p\nabla G)\,\mathrm{d}V &= \int_{S+\mathcal{B}_D}(G\nabla p - p\nabla G)\cdot\boldsymbol{n}\,\mathrm{d}S\\ &= -\int_S (G\nabla p - p\nabla G)\cdot\boldsymbol{n}\,\mathrm{d}S\\ &\qquad + \int_{\mathcal{B}_D}(G\nabla p - p\nabla G)\cdot\boldsymbol{n}\,\mathrm{d}S\,. \end{aligned}\tag{17}$$

Note that the signs of the terms the right of Eq. (17) are opposite, since their normal vectors are opposed. We've chosen the negative orientation (unit vectors oriented *inward*) for convenience. However we can get rid of this last term altogether if we argue that G and p are small on the outer surface $\mathcal{B}_D$. This should be the case, as long as D is large—and we haven't yet made any assumptions about D—so we can safely neglect the contribution on the outer surface.[1] The gradient of a function dotted with the normal vector (i.e., the directional derivative) is often called the normal derivative, written

$$\nabla f\cdot\boldsymbol{n} = \frac{\partial f}{\partial n}\,.\tag{18}$$

So, with all these simplifications, and using the Green's function definition, we can write Eq. (15) as

$$\begin{aligned}-4\pi p(\boldsymbol{r}) &= -\int_S\left[\frac{e^{ikR}}{R}\frac{\partial p}{\partial n} - p\frac{\partial}{\partial n}\left(\frac{e^{ikR}}{R}\right)\right]\mathrm{d}S\,,\qquad\text{or}\\ p(\boldsymbol{r}) &= \frac{1}{4\pi}\int_S\left[\frac{e^{ikR}}{R}\frac{\partial p}{\partial n} - p\frac{\partial}{\partial n}\left(\frac{e^{ikR}}{R}\right)\right]\mathrm{d}S\,.\end{aligned}\tag{19}$$

Equation (19) is the ***Kirchoff-Helmholtz Integral***, and states that we can find the pressure field external to a surface which bounds an arbitrary distribution of sources, provided that we know the pressure and its normal derivative on that surface.

4.2 Rayleigh Integral

If we imagine we have some flat, rigid surface at $z = 0$, it might be simpler to choose a Green's function that matches the boundary conditions rather than enforcing them in the solution. In the case of a rigid boundary, we have and image at $\boldsymbol{r}_i$

$$G = \frac{e^{ikR}}{R} + \frac{e^{ikR'}}{R'}\tag{20}$$

[1] More precisely, this statement invokes the Sommerfield radiation condition, which requires that G and p must decay at least as fast as r^{-1}. This is true for just the geometric spreading of both, but will be even more comfortably satisfied if we include any sort of attenuation (which would exist in any real propagation).

Then, since $\boldsymbol{R} \cdot \boldsymbol{e}_z = -\boldsymbol{R}_i \cdot \boldsymbol{e}_z$, we can see that $\partial G/\partial n$ will vanish along $z = 0$. Therefore the second term in Eq. (19) vanishes when S coincides with S. Then, note that

$$\frac{\partial p}{\partial n} = \nabla p \cdot \boldsymbol{n} = \rho_0 \frac{\partial \boldsymbol{u}}{\partial t} \cdot \boldsymbol{n} \tag{21}$$

Substituting this into the first term of Eq. (19) gives the *Rayleigh integral*:

$$p(\boldsymbol{r}, t) = \frac{\rho_0}{4\pi} \int_S \frac{e^{ikR}}{R} \dot{\boldsymbol{u}}(\boldsymbol{r}') \cdot \boldsymbol{e}_n \, \mathrm{d}S \,. \tag{22}$$

It gives an expression for the pressure due to an arbitrary source as a summation of the contributions from the source The primed coordinates refer to coordinates on the source surface, and the distance $R = |\boldsymbol{R}|$, where

$$\boldsymbol{R} \equiv \boldsymbol{r} - \boldsymbol{r}' \,. \tag{23}$$

4.3 The Integration

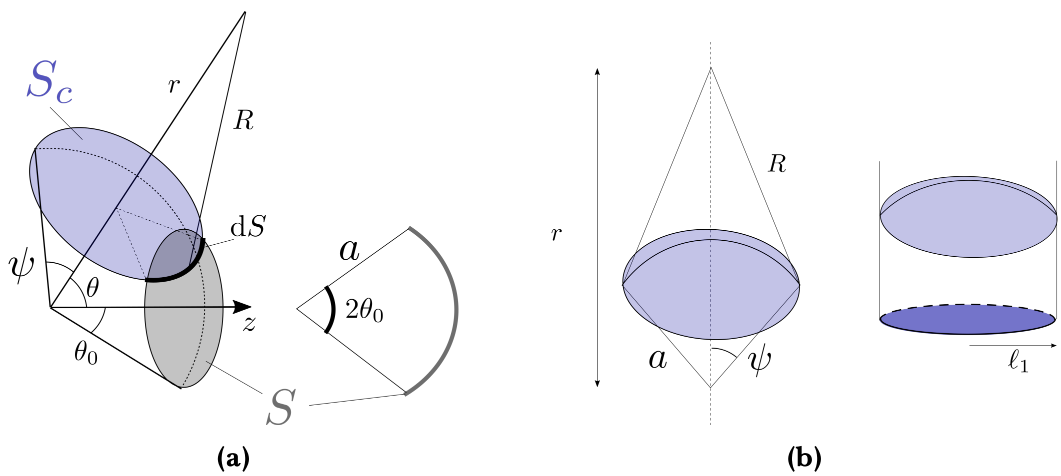


Figure 7: Integration geometry for Eq. (4) with a spherical transducer. **(a)** Dimensions for arbitrary field point. **(b)** Area of the spherical cap S_C.

It turns out the convenient coordinate to integral over is R, the distance from the point on the transducer $\boldsymbol{r}'$ to the field point $\boldsymbol{r}$; see Fig. 7a. To find the radiating element $\mathrm{d}S$, consider the hemispherical cap S_C. How do we find its area? We could do the integral, or note as Archimedes did, that if we project it to a circle with radius ℓ_1, we have the same area; see Fig. 7b So, $A = \pi \ell_1^2$. What is ℓ_1 though? From the geometry,

$$\ell_1 = \sqrt{\frac{a}{r}} \, \left[R^2 - (r-a)^2 \right]^{\frac{1}{2}} \tag{24}$$

Thus the area S is

$$S_c = \pi \ell_1^2 = \pi \, \frac{a}{r} \left[R^2 - (r-a)^2 \right] \, , \tag{25}$$

thus the differential element is

$$\mathrm{d}S = 2\pi \frac{a}{r} R \, \mathrm{d}R \, . \tag{26}$$

What fraction of this intersects the transducer region? It can be shown that this is

$$\alpha(\psi) = \frac{1}{\pi} \arccos \left[\frac{\cos\theta_0 - \cos\psi \cos\theta}{\sin\psi \sin\theta} \right], \tag{27}$$

where

$$\cos(\psi) = \frac{r^2 + a^2 - R^2}{2ar} \, . \tag{28}$$

Thus the total integration is

$$p(\boldsymbol{r}) = \frac{-i\omega\rho_0 u_0 a}{r} \int_{R_{\min}}^{R_{\max}} \alpha(\psi) \, e^{ikR} \, \mathrm{d}R \, . \tag{29}$$

References

[1] Allan D. Pierce. *Acoustics: an Introduction to Its Physical Principles and Applications.* Acoustical Society of America, Melville, NY, 1989.