Math 6701 Exam 2 Review

Matrices

The **eigenvalues** λ of a square matrix $A \in \mathbb{R}^{n\times n}$ are found by solving the characteristic equation for λ

$$\det(A - \lambda \mathbf{1}) = 0\,. \tag{1}$$

The corresponding **eigenvectors** $\boldsymbol{v}$ satisfy

$$(A - \lambda\mathbf{1})\,\boldsymbol{v} = \mathbf{0}\,. \tag{2}$$

If we have n linearly independent eigenvectors, then the matrix may be diagonalized as

$$A = PDP^{-1}\,, \tag{3}$$

where P is the matrix with the eigenvectors of A as columns and

$$D = \begin{pmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & & \lambda_n \end{pmatrix}\,, \tag{4}$$

Symmetric matrices correspond to quadratic forms. Note that

$$\begin{aligned} (x_1, x_2)\begin{pmatrix} a & b \\ b & c\end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= k^2 \\ \longrightarrow a^2x_1^2 + 2bx_1x_2 + cx_2^2 &= k_2. \end{aligned} \tag{5}$$

So if we write $\boldsymbol{y} = P^{-1}\boldsymbol{x}$, where P is the matrix with the *normalized* eigenvectors of $A = PDP^{-1}$ as comlumns then

$$\begin{aligned} \boldsymbol{x}^\top A\boldsymbol{x} &= (\boldsymbol{y})^\top \left(PDP^{-1}\right)(P\boldsymbol{y}) \\ &= \boldsymbol{y}^\top P^\top PDP^{-1}P\boldsymbol{y} \\ &= \boldsymbol{y}^\top D\boldsymbol{y}\,. \end{aligned} \tag{6}$$

The multiplication then gives

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 = k^2\,, \tag{7}$$

which is an ellipse, hyperbola, circle, or line, depending on the values of λ.

To find the **least squares** fit to a linear model (meaning the *coefficients* are linear), write error as $\|A\boldsymbol{x} - \boldsymbol{b}\|$. Here $\boldsymbol{b}$ are the data points for each t, and A is the model evaluated at each t. For example, for a quadratic fit, let $y = c\,t^2 + d\,t + f$. Then,

$$A = \begin{pmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ \vdots & \vdots & \vdots \\ t_n^2 & t_n & 1 \end{pmatrix}\,, \quad \boldsymbol{x} = \begin{pmatrix} c \\ d \\ f \end{pmatrix}\,, \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}\,. \tag{8}$$

We need to find $\boldsymbol{x}$ to minimize

$$\begin{aligned} \epsilon &= \frac{1}{2}\|A\boldsymbol{x} - \boldsymbol{b}\|^2 = \frac{1}{2}(A\boldsymbol{x} - \boldsymbol{b})^\top(A\boldsymbol{x} - \boldsymbol{b}) \\ \nabla\epsilon &= A^\top A\boldsymbol{x} - A^\top\boldsymbol{b} \\ &\longrightarrow \quad \text{solve} \quad A^\top A\boldsymbol{x} - A^\top \boldsymbol{b} = \mathbf{0}\,. \end{aligned} \tag{9}$$

Since the columns of A are linearly independent, $\boldsymbol{x}$ is just the projection of $\boldsymbol{b}$ onto span $\{\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \boldsymbol{a}_3\}$. To find the projection, use an orthonormal basis found from the Gram-Schmidt process. Equation (9) is directly solvable, provided $(A^\top A)^{-1}$ exists.

If A has linearly independent columns, $A^\top A$ is invertible, since

$$\begin{aligned} &A\boldsymbol{v} = \mathbf{0} \text{ iff } \boldsymbol{v} = \mathbf{0} \text{ (columns of } A \text{ are L.I.)} \\ &A^\top A\boldsymbol{u} = \mathbf{0} \implies \boldsymbol{u} \in \mathcal{N}(A) \\ &\boldsymbol{u}^\top A^\top A\boldsymbol{u} = \mathbf{0} = (A\boldsymbol{u})\cdot(A\boldsymbol{u}) = \|A\boldsymbol{u}\|^2 = 0 \text{ iff } \boldsymbol{u} = \mathbf{0}\,. \end{aligned} \tag{10}$$

So the columns of $(A^\top A)$ are L.I. and thus it's invertible.

The **matrix exponential** may be computed from

$$\begin{aligned} e^A &= \sum_{n=0}^{\infty}\frac{A^n}{n!} \\ &= \mathbf{1} + A + \frac{1}{2}A^2 + \dots \end{aligned} \tag{11}$$

If A is diagonalizable, we can write

$$\begin{aligned} e^A &= P\left(\mathbf{1} + D + \frac{1}{2}D^2 + \dots\right)P^{-1} \\ &= P\begin{pmatrix} e^{\lambda_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & & e^{\lambda_n} \end{pmatrix}P^{-1}\,. \end{aligned} \tag{12}$$

Differential Equations

The **existence and uniqueness theorem** says that the initial value problem

$$a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \dots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x) = f(x) \tag{13}$$

with initial values specfified for $y(x_0)$, $y''(x_0)$, ..., $y^{(n-1)}(x_0)$, there exists a unique solution provided that all the $a_i(x)$ and $f(x)$ are continuous on an interval containing x_0.

The full solution to equations of the type

$$mx'' + cx' + kx = f(t)\,, \tag{14}$$

is the sum of the homogeneous solution x_h [which solves Eq. (14) when $f(t) = 0$] and the particular solution x_p. To find the homogeneous part, we find the roots λ_i of the **characteristic equation**, and then

$$x_h = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}\,. \tag{15}$$

Depending on the sign of $\mathcal{D} = c^2 - 4km$, we can have a few types of solutions, overdamped ($\mathcal{D} > 0$), underdamped ($\mathcal{D} < 0$) or critically damped ($\mathcal{D} = 0$).

Discriminant	Solution Form
$\mathcal{D} > 0$	$x_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$
$\mathcal{D} = 0$	$x_h(t) = e^{-ct/2m}\,(C_1 + C_2 t)$
$\mathcal{D} < 0$	$x_h(t) = e^{-ct/2m}\,(C_1\cos\omega_0 t + C_2 \sin\omega_0 t)$

The particular solution for the forcing function is found by guessing a particular solution with the same form as $f(t)$. Note that *if the particluar form already contains the guessed form*, we must first multiply by t. For example,

$$x'' + 9x = \cos 3t\,, \tag{16}$$

our guess for the particular part must be multiplied by t, i.e., $x_p = At\cos 3t + Bt\sin 3t$.

For **systems of ODEs** we have the general form

$$\boldsymbol{X}' = A\boldsymbol{X} + \boldsymbol{b}(t)\,, \tag{17}$$

where $\boldsymbol{X} = [x_1(t), \dots, x_n(t)]^\top$. To solve Eq. (17), we first find the eigenvalues and vectors [cf. Eqs. (1) and (2)]. If we have n *linearly independent* eigenvectors, we can immediately write the solution to the homogeneous part

$$\boldsymbol{X}_h = C_1\boldsymbol{v}_1 e^{\lambda_1 t} + C_2\boldsymbol{v}_2 e^{\lambda_2 t} + \dots\,. \tag{18}$$

If don't have enough linearly independent eigenvectors, then we will need to find additional solutions from the generalized eigenvectors. That is, we solve in sequence

$$\begin{aligned} (A - \lambda\mathbf{1})\,\boldsymbol{v} &= \mathbf{0} \\ (A - \lambda\mathbf{1})\,\boldsymbol{w} &= \boldsymbol{v} \\ (A - \lambda\mathbf{1})\,\boldsymbol{z} &= \boldsymbol{w} \\ &\text{etc.} \end{aligned}$$

up to the multiplicity of λ. Finally, the solutions for that eigenvalue are

$$\begin{aligned} \boldsymbol{x}_1 &= e^{\lambda t}\boldsymbol{v} \\ \boldsymbol{x}_2 &= e^{\lambda t}(t\boldsymbol{v} + \boldsymbol{w}) \\ \boldsymbol{x}_3 &= e^{\lambda t}\left(\frac{t^2}{2}\boldsymbol{v} + t\boldsymbol{w} + \boldsymbol{z}\right)\,. \end{aligned}$$

Most generally, the solution to Eq. (17) is

$$\boldsymbol{X} = e^{A(t-t_0)}\boldsymbol{X}_0 + e^{At}\int_{t_0}^{t} e^{-As}\boldsymbol{b}(s)\,\mathrm{d}s\,. \tag{19}$$

To find the particular solution to Eq. (17), will almost certainly be able to use variation of parameters. That is, e.g., if $\boldsymbol{b} = \left[1+t, 3, t^2\right]^\top$, we would guess a particular solution

$$\boldsymbol{X}_p = \begin{pmatrix} a_0 + a_1 t + a_2 t^2 \\ b_0 + b_1 t + b_2 t^2 \\ c_0 + c_1 t + c_2 t^2 \end{pmatrix} = \begin{pmatrix} a_0 \\ b_0 \\ c_0 \end{pmatrix} + \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} t + \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} t^2$$