Math 6701 Exam 1 Review

Vector Spaces

Call by $\mathbb{R}^n$ the set of n-tuples of real numbers. Elements of these sets are called **vectors**, e.g.,

$$\boldsymbol{x} = (x_1,\, x_2) \qquad \in \mathbb{R}^2\,,$$
$$\boldsymbol{x} = (x_1,\, x_2,\, x_3) \in \mathbb{R}^3\,, \quad \text{etc.},$$

where $x_i \in \mathbb{R}$. The magnitude of a vector $\boldsymbol{x} \in \mathbb{R}^n$ is defined by

$$\|\boldsymbol{x}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}\,. \tag{1}$$

A unit vector has magnitude 1. A unit vector in the direction of $\boldsymbol{x}$ is

$$\hat{\boldsymbol{x}} = \boldsymbol{x}/\|\boldsymbol{x}\|\,. \tag{2}$$

Vector Operations

Define scalar multiplication

$$\lambda\boldsymbol{x} = \lambda\,(x_1, \ldots,\, x_n) = (\lambda x_1, \ldots,\, \lambda x_n)\,, \tag{3}$$

and vector addition

$$\boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \ldots,\, x_n + y_n)\,. \tag{4}$$

A space containing vectors in $\mathbb{R}^n$ and which is closed under scalar multiplication and vector addition is called a **vector space**. A space S is called a **subspace** of V ($S \subseteq V$) if S is itself a vector space and $\boldsymbol{s} \in V\ \forall \boldsymbol{s} \in S$ (note $V \subseteq V$).

Properties of Vector Spaces

For some vector space V containing $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$, the following hold:

1. Commutivity $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$
2. Associativity $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$
3. Multiplicative Distribution $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y}$
4. Multiplicative Association $(\alpha\beta)\boldsymbol{x} = \alpha(\beta\boldsymbol{x}) = \beta(\alpha\boldsymbol{x})$
5. Additive Identity $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{x}$
6. Multiplicative Identities $1\boldsymbol{x} = \boldsymbol{x}$ and $0\boldsymbol{x} = \boldsymbol{0}$

Note that vector spaces *must contain the zero vector*!

Dot Product

The dot product of vectors $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$ is defined by

$$\boldsymbol{x} \cdot \boldsymbol{y} \equiv \sum_{i=1}^{n} x_n y_n = \|\boldsymbol{x}\|\|\boldsymbol{y}\| \cos\theta\,, \tag{5}$$

where θ is the angle between them. To show this, define $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$, then from the law of cosines

$$\|\boldsymbol{z}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\|\boldsymbol{x}\|\|\boldsymbol{y}\| \cos\theta\,, \tag{6}$$

But

$$\begin{aligned} \|\boldsymbol{z}\|^2 &= \boldsymbol{z} \cdot \boldsymbol{z} = (\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \\ &= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\boldsymbol{x} \cdot \boldsymbol{y}\,. \end{aligned} \tag{7}$$

Comparing Eqs. (6) and (7) implies Eq. (5).

Cross Product

The cross product is a *vector* product of two vectors $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^3$. It can be computed from the determinant

$$\begin{aligned} \boldsymbol{x} \times \boldsymbol{y} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} &= (x_2 y_3 - y_2 x_3), \boldsymbol{e}_x - \\ &(x_1 y_3 - y_1 x_3), \boldsymbol{e}_x + (x_1 y_2 - y_1 x_2), \boldsymbol{e}_z\,. \end{aligned} \tag{8}$$

It turns out that

$$\|\boldsymbol{x} \times \boldsymbol{y}\| = \|\boldsymbol{x}\|\|\boldsymbol{y}\| \sin\theta\,. \tag{9}$$

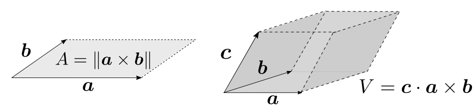
This proof is a bit longer, but to show this, use Lagrange's identity

$$\|\boldsymbol{x} \times \boldsymbol{y}\|^2 = (\|\boldsymbol{x}\|\|\boldsymbol{y}\|)^2 - (\boldsymbol{x} \cdot \boldsymbol{y})^2\,. \tag{10}$$

Then use Eq. (5) and $\cos^2\theta = 1 - \sin^2\theta$ to complete the proof. Also note that the cross product is *not associative*, and that

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c}\,(\boldsymbol{a} \cdot \boldsymbol{b}) - \boldsymbol{b}\,(\boldsymbol{a} \cdot \boldsymbol{c})\,. \tag{11}$$

Dot and cross products can be used to find areas and volumes



Vector Equations

Suppose we have two points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, and define the vector $\boldsymbol{a} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ the line that connects them is

$$\boldsymbol{\ell}(t) = \boldsymbol{x}_1 + \boldsymbol{a}t = \boldsymbol{x}_2 + \boldsymbol{a}t\,. \tag{12}$$

To find shortest distance between a point and a line/plane, define $r = \|\boldsymbol{r}\|$ and minimize $\partial r/\partial x$, $\partial r/\partial y$, etc. The span of two vectors is a plane which is subspace of $\mathbb{R}^n$

Linear Independence

A set of n vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in V$ is **linearly independent** if and only if the *only* solution to

$$c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_n = \boldsymbol{0} \tag{13}$$

is $c_1 = c_2 = \ldots = c_n = 0$. Note that a set of vectors containing the zero vector *cannot* be linearly independent.

Bases

The **span** of a set of n vectors is all linear combinations thereof:

$$\operatorname{span}\{\boldsymbol{v}_i\} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_n\,. \tag{14}$$

If $\boldsymbol{v}_i$ are linearly independent, then they span $\mathbb{R}^n$. In this case we have a **basis**. Note that the vectors need not be orthogonal. If we have an orthonormal basis of a vector space v with unit vectors $\hat{\boldsymbol{v}}_i$, then the projection of $\boldsymbol{x}$ onto the subspace V is

$$\operatorname{proj}_V \boldsymbol{x} = (\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_1)\,\hat{\boldsymbol{v}}_1 + (\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_2)\,\hat{\boldsymbol{v}}_2 + \ldots + (\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_n)\,\hat{\boldsymbol{v}}_n\,. \tag{15}$$

Continuous Functions

Principles of vector spaces can be applied to continuous functions. For example, $\mathbb{P}_n$—the set of polynomials of degree n—is a vector space with basis 1, x, …, x^n. Analogous to the dot product, we define the inner product of two functions $f(x)$ and $g(x)$ on some interval $[a,\, b]$ as

$$(f,\, g) \equiv \int_a^b f(x)\, g(x)\, \mathrm{d}x\,. \tag{16}$$

The "length" of a function is

$$\|f(x)\| \equiv \left[\int_a^b f^2(x)\, \mathrm{d}x\right]^{\frac{1}{2}}\,, \tag{17}$$

which requires that the function is *square integrable* [i.e., Eq. (17) is bounded].

Gram–Schmidt Process

Suppose we have a set of n linearly independent vectors $\boldsymbol{v}_i \in \mathbb{R}^n$. These vectors form a basis, but what if we want an orthonormal basis $\boldsymbol{n}_i$, such that $\boldsymbol{n}_i \cdot \boldsymbol{n}_j = \delta_{ij}$? Follow these steps

1. Normalize the first vector $\boldsymbol{n}_1 = \boldsymbol{v}_1/\|\boldsymbol{v}_1\|$
2. Create intermediate vector $\boldsymbol{y}_2$ from $\boldsymbol{v}_2$ by subtracting its component along $\boldsymbol{n}_1$
$$\boldsymbol{y}_2 = \boldsymbol{v}_2 - (\boldsymbol{v}_2 \cdot \boldsymbol{n}_1)\ \boldsymbol{n}_1\,.$$
3. Normalize $\boldsymbol{n}_2 = \boldsymbol{y}_2/\|\boldsymbol{y}_2\|$.
4. Repeat, subtracting out all previous components; for example
$$\boldsymbol{y}_3 = \boldsymbol{v}_3 - (\boldsymbol{v}_3 \cdot \boldsymbol{n}_1)\ \boldsymbol{n}_1 - (\boldsymbol{v}_3 \cdot \boldsymbol{n}_2)\ \boldsymbol{n}_2\,.$$

Matrices

An m-by-n matrix A has m rows and n columns, and is written $A \in \mathbb{R}^{m\times n}$ (or $\mathbb{C}^{m\times n}$ if its entries are complex).

Associated Spaces

1. **Column Space** $\mathcal{R}(A)$: span of the columns of A
2. **Null Space** $\mathcal{N}(A)$: the set $\{\boldsymbol{v}\}$ such that $A\boldsymbol{v} = \boldsymbol{0}$.
3. **Row Space** $\mathcal{R}(A^\mathsf{T})$: span of the rows of A

The **rank** of matrix is the dimension of the vector space spanned by its rows. This the same as the dimension of its column space. The **nullity** of matrix is the dimension of its null space. For $A \in \mathbb{R}^{m\times n}$,

$$\operatorname{rank} A + \operatorname{null} A = n\,. \tag{18}$$

Matrix Manipulation

Rows of a matrix A may be exchanged, added, or multiplied by a constant without changing the span of its rows. A matrix A is in **row echelon form** when all pivot entries are equal to 1 and strictly to the right of pivot entries in the rows above.

1. The rows of $\operatorname{rref} A$ form a basis for $\mathcal{R}(A^\mathsf{T})$
2. The pivot columns of $\operatorname{rref} A$ are the columns of A that form a basis for $\mathcal{R}(A)$