ME 8873 - Georgia Tech

Lecture 18: Trans-Skull Focusing

5 March 2020

Lecturer: Scott Schoen Jr

0 The Problem

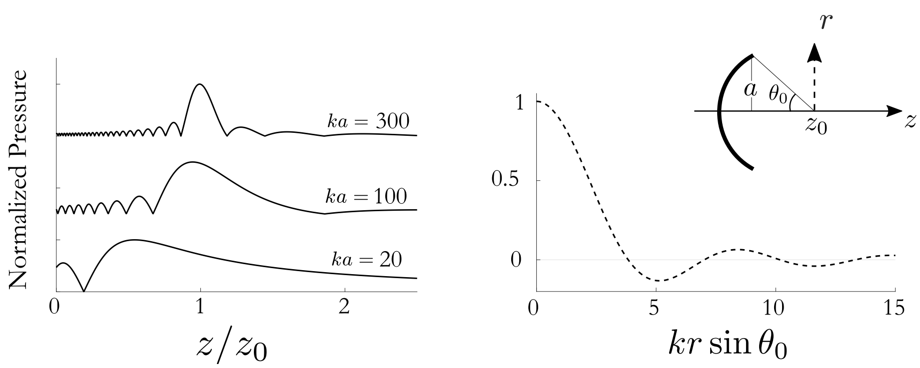


Figure 1: (Left) Normalized axial pressure for focused ultrasound transducers with the indicated values of ka. (Right) Normalized transverse pressure in the focal plane.

Focused ultrasound transducers allow localization of the acoustic energy at a desired point as in Fig. 1. This allows us to, for example, deposit thermal energy for ablation as discussed last lecture. However, the focusing relies on constructive interference; a change in the acoustic impedance of the medium, then the resulting focus may be destroyed (see Fig. 2).

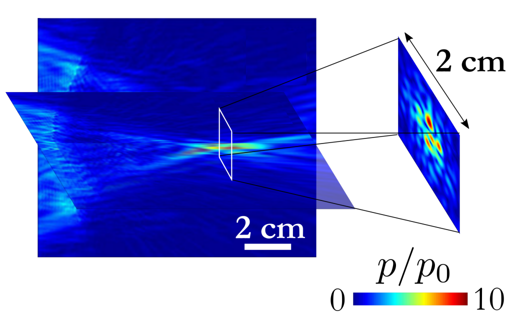


Figure 2: Simulation of aberration of the focal field of a $1\,\mathrm{MHz}$ transducer ($230\,\mathrm{mm}$ diameter, $F_{\#} = 1$) due to propagation through the skull. Adapted from Fig. 9 of Ref. 1.

Example 1: Trans-skull Challenges

Why is the focus destroyed?
Ans: Mainly refraction. The change in acoustic impedance between different media causes the path (and thus the phase) of the waves, and they no longer interfere constructively at the focus.

Other Challenges?
Reflection due to impedance mismatch, absorption causing heating.

1 Focusing with Phased Arrays

1.1 Homogeneous Approach

A spherical transducer ensures that the sound leaves the surface such that the distance to the focal spot is the same for any part of the surface. But, this means if we want to change the focal position, we have to physically move the transducer.

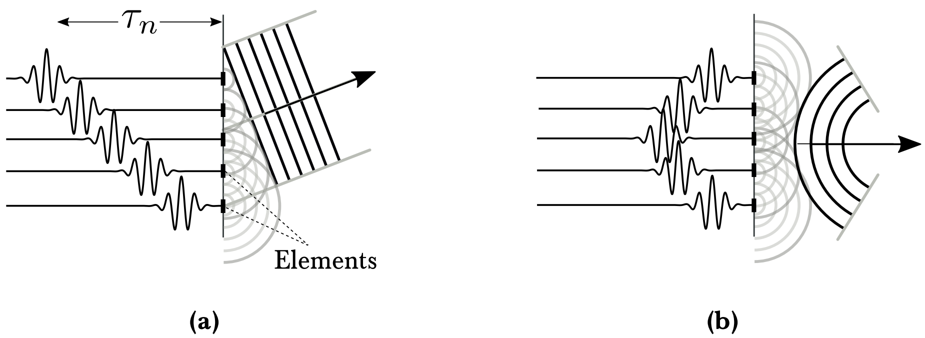


Figure 3: **(a)** Steering and **(b)** focusing with a phased array.

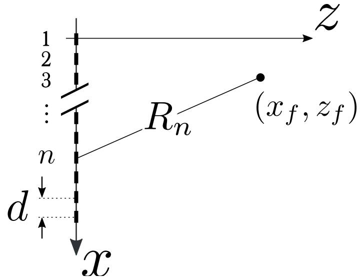
However, suppose we have an array of individual elements we can control individually. If we are able to control the time delay of each element, then we can steer the emitted radiation (Fig. 3a) or achieve focusing (Fig. 3b). A huge advantage of phased arrays is that we can focus to arbitrary locations, not necessarily just along the axis.

Example 2: Focusing at Arbitrary Point

What are the time delays to focus at are arbitrary point (x_f, z_f)?
Ans: Well, we want all the waves to arrive in phase (i.e., at the same time). So the delays will have to be equal to the travel time from each transmitter to the

Example 2 (*cont.*)

desired focus. Call the plane of the transducer the x-axis, and the perpendicular direction the z.

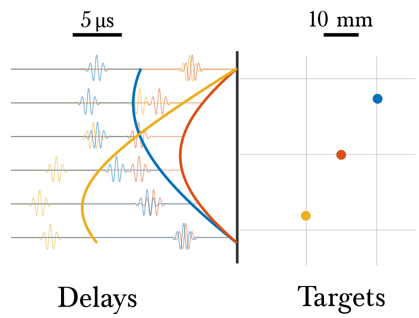


The distance to the focal point for the n^{th} transmitter is

$$R_n = \sqrt{[x_f - (n-1)d]^2 + z_f^2}\,.$$

Then, the n^{th} delay will be given by

$$\tau_n = R_n / c_0\,.$$



1.2 Through a Layer

Consider the model problem of two sources, whose fields we wish to have constructively interfere at the focal point as in Fig. 4. Two sources of the same frequency ω are separated by distance ℓ and want to focus through a layer of thickness d. What should be the time delays so that the rays arrive in phase?

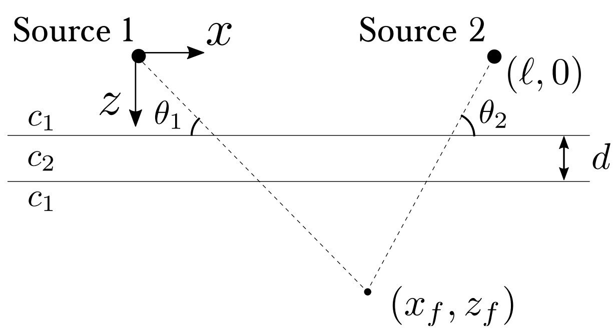


Figure 4: Focusing through a layer

Looking at each source, the total path length along each of the dotted lines is[†]

$$r_i = \frac{z_f}{\sin\theta_i}\,. \tag{1}$$

The length of the path within the layer is

$$r_{\text{layer},i} = \frac{d}{\sin\theta_i}\,. \tag{2}$$

So the time to travel along the line is

$$\begin{aligned}
t_i &= \Big(\text{Time in } c_1\Big) + \Big(\text{Time in } c_2\Big) \\
&= \frac{1}{c_1}\left(r_i - r_{\text{layer},i}\right) + \frac{1}{c_2} r_{\text{layer},i} \\
&= \frac{1}{c_1}\left(\frac{z_f}{\sin\theta_i} - \frac{d}{\sin\theta_i}\right) + \frac{1}{c_2}\frac{d}{\sin\theta_i} \\
&= \frac{1}{c_1}\frac{z_f}{\sin\theta_i} + \frac{d}{\sin\theta_i}\left(\frac{1}{c_2} - \frac{1}{c_1}\right)
\end{aligned} \tag{3}$$

Example 3: Limiting Cases of Eq. (3)

What is the Time With No Layer?
Ans: $t_i = (z_f/c_1)/\sin\theta_i$.

What if Layer has $c_2 = c_1$?
Ans: Then second term vanishes, left with no layer case.

Now, what's the time delay between the two so that they arrive in phase? Well, consider the time delay $\Delta t = t_1 - t_2$, or

$$\Delta t = \left[\frac{1}{c_1}\frac{z_f}{\sin\theta_1} + \frac{d}{\sin\theta_1}\left(\frac{1}{c_2} - \frac{1}{c_1}\right)\right] - \left[\frac{1}{c_1}\frac{z_f}{\sin\theta_2} + \frac{d}{\sin\theta_2}\left(\frac{1}{c_2} - \frac{1}{c_1}\right)\right] \tag{4}$$

With some algebra (see Sec. 4.1) gives

$$\Delta t = \left[\frac{z_f}{c_1} + d\left(\frac{1}{c_2} - \frac{1}{c_1}\right)\right]\left(\frac{1}{\sin\theta_1} - \frac{1}{\sin\theta_2}\right)\,. \tag{5}$$

In the case where the sources are equidistant, then the term in parentheses vanishes as expected. More generally, we can just draw a line along the path from the receiver to the desired source position, and sum up the time it takes to get through each position

$$t_i = \sum \Delta t_n = \sum \frac{s_i}{c_i}\,. \tag{6}$$

[†]Obviously we're making some big assumptions here. We'll address this later.

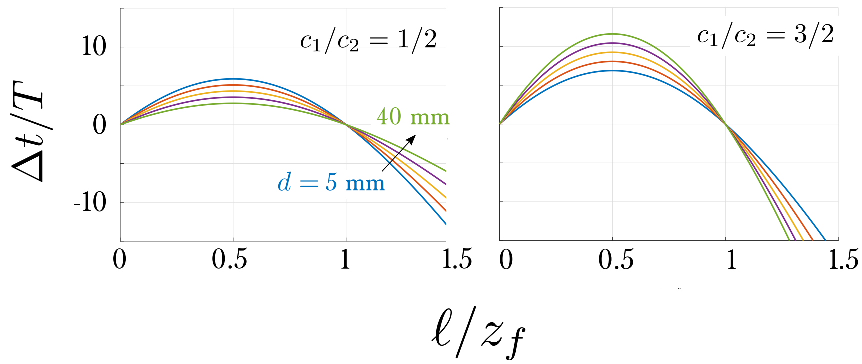


Figure 5: Time delays (normalized by the period for a 2 MHz source) for the indicated layer thickness, as a function of the source position for a hard layer (left) and soft layer (right).

Example 4: Limitations of Ray-based Focusing

What are we ignoring with this approach?
Ans: Reflection, refraction. Ray-approximation is valid for $kd \gg 1$. Also assumes layer is homogeneous.

What happens if $\Delta t/T = 0.5$?
The waves will cancel out! So need to be accurate with delays.

1.3 Including Refraction

Since the medium above and below are the same, the exit angle will be the same as the entrance angle.† The ray will be refracted according to Snell's law

$$\frac{\sin\left(\frac{\pi}{2} - \theta_i\right)}{c_1} = \frac{\sin\theta_\ell}{c_2} \implies \sin\theta_\ell = \frac{c_2}{c_1}\cos\theta_i \tag{7}$$

where θ_ℓ is the angle the ray makes with the normal in the layer (note that it is cosine due to how we have defined θ_1). Its path length in the layer is then

$$r_{\text{layer},i} = \frac{d}{\cos\theta_\ell} = \frac{d}{\sqrt{1-\sin^2\theta_\ell}} = \frac{d}{\sqrt{1-(c_2/c_1)^2\cos^2\theta_i}}. \tag{8}$$

†Consider that Snell's law says $\sin\theta_1/c_1 = \sin\theta_2/c_2$ and $\sin\theta_2/c_2 = \sin\theta_3/c_3$. Then since $c_1 = c_3$, it follows that $\theta_1 = \theta_3$.

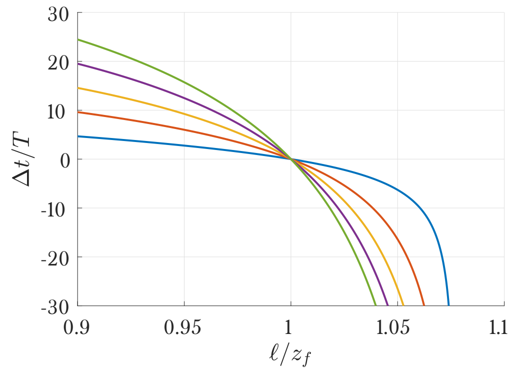


Figure 6: Time delays (normalized by the period for a 2 MHz source) for 5 to 40 mm layer thickness (as in Fig. 5), including refraction.

Compare with Eq. (2). Then, the total time is

$$t_i = \frac{1}{c_1}\frac{z_f}{\sin\theta_i} + \frac{d}{\sqrt{1-(c_2/c_1)^2\cos^2\theta_i}}\left(\frac{1}{c_2}-\frac{1}{c_1}\right). \tag{9}$$

Accounting for refraction muddies up the expressions. From Eq. (9), we see that we may now have a critical angle for a hard layer (i.e., $c_2 > c_1$), in which case the wave will not go through at all; see Fig. 6 where $\Delta t \to -\infty$ since the second ray never arrives. But wait, the rays will bend as they travel through the layer, so our distances will be thrown off right? Yes, so you can appreciate the complexity of the problem (and hence why only small variances in ℓ are shown in Fig. 6.

2 Spectral Method

The methods discussed in the model problem operate in the time domain. However, the approach[2] used in clinical practice is an elaboration of the point technique, but its computations are performed in the spatial frequency domain.

2.1 Linearity of the Wave Equation

The wave equation has the nice property that it is linear. To show this, recall that by definition, an operator $\mathcal{L}$ is linear if and only if

$$\mathcal{L}[\alpha f + \beta g] = \alpha\mathcal{L}[f] + \beta\mathcal{L}[g]\,. \tag{10}$$

for any scalar constants $\alpha, \beta \in \mathbb{R}$. For the wave equation, $\mathcal{L} = (\nabla^2 + c_0^{-2}\partial^2/\partial t^2)$, so we have

$$\begin{aligned}\left(\nabla^2 + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)[\alpha f + \beta g] &= \left[\nabla^2(\alpha f) + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}(\alpha f)\right] + \left[\nabla^2(\beta g) + \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}(\beta g)\right] \\ &= \alpha\left(\nabla^2 f + \frac{1}{c_0^2}\frac{\partial^2 f}{\partial t^2}\right) + \beta\left(\nabla^2 g + \frac{1}{c_0^2}\frac{\partial^2 g}{\partial t^2}\right) \\ &\overset{\checkmark}{=} \alpha\mathcal{L}[f] + \beta\mathcal{L}[g]\,. \end{aligned} \tag{11}$$

Now if f and g are solutions of the wave equation, then $\mathcal{L}[f] = \mathcal{L}[g] = 0$, and thus any linear combination of f and g is also a solution. We will rely on linearity to justify out consideration of single-frequency signals.

2.2 Temporal Frequency

Suppose our pressure time series has only one frequency, that is

$$\begin{aligned} p(\boldsymbol{r}, t) &= \tilde{p}(\boldsymbol{r}) \cdot \cos\omega t \\ &= \mathrm{Re}\left[\tilde{p}(\boldsymbol{r}) \cdot e^{-i\omega t}\right], \end{aligned} \tag{12}$$

where we've chosen the negative sign convention (see Sec. 4.2). From now on, we won't write Re everywhere; we'll take it as implied that the pressure that we would measure is the real part of the complex quantity $\tilde{p}$. Then the wave equation becomes

$$\begin{aligned} \nabla^2\left(\tilde{p}e^{-i\omega t}\right) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\left(\tilde{p}e^{-i\omega t}\right) &= 0 \\ e^{-i\omega t}\nabla^2\tilde{p} - \frac{1}{c_0^2}\tilde{p}\frac{\partial^2}{\partial t^2}\left(e^{-i\omega t}\right) &= 0 \\ \left[\nabla^2\tilde{p} - \frac{(-i\omega)^2}{c_0^2}\tilde{p}\right]e^{-i\omega t} &= 0 \quad\Longrightarrow\quad \nabla^2\tilde{p} + \frac{\omega^2}{c_0^2}\tilde{p} = \left(\nabla^2 + k^2\right)\tilde{p} = 0\,. \end{aligned} \tag{13}$$

Equation (13) is called the Helmholtz equation (and it's also linear). This has the advantage of now we only have a spatial derivative, instead of time and space derivatives like the full wave equation. But, this only works for single frequencies, so why is this useful?

Recall the Fourier transform, which is defined

$$\mathcal{F}[\,\cdot\,] = \int_{-\infty}^{\infty} (\,\cdot\,)\, e^{i\omega t}\,\mathrm{d}t\,. \tag{14}$$

Qualitatively, it gives the amplitude and phase associated with each frequency ω, such that if we add up sine waves at each frequency with that amplitude and phase, we'll get the original

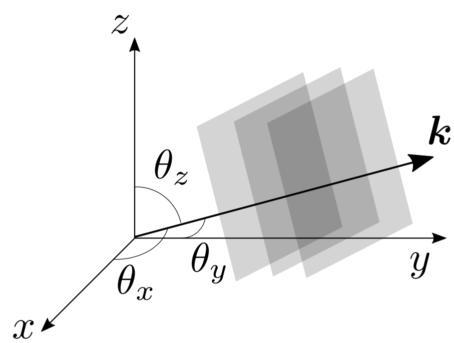


Figure 7: Plane wave and associated directional angles.

signal.

Now notice the Fourier transform is also linear. So, by solving the Helmholtz equation for each frequency component, we can can solve it for arbitrary time series. We break up $p(t)$ into its components, solve the Helmholtz equation instead, add up the component solutions and take the inverse transform. Why go to all this trouble? To make the math easier!

2.3 Spatial Frequency

Now further suppose that our single frequency pressure field is a plane wave in 3D, which we can write as write

$$\begin{aligned} \tilde{p}(\boldsymbol{r}) &= p_0\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \\ &= p_0\, e^{i(k_x x + k_y y + k_z z)} \end{aligned} \tag{15}$$

(note we'll stop writing $e^{-i\omega t}$ since everything is proportional to it, so we'll take it as implied from now on). The wavenumber components k_i define the angle with which the wave travels, see Fig. 7. But for simplicity, consider a plane wave traveling along the x-direction, such that

$$\tilde{p} = p_0\, e^{ikx} \tag{16}$$

We say the field oscillates in time with frequency ω (i.e., $\omega/2\pi$ times every second, the field varies between $\|\tilde{p}\|$ and $-\|\tilde{p}\|$). Similarly, we can say it varies in space with spatial frequency $1/\lambda$. Just like with frequency, it's convenient to define an angular frequency $2\pi/\lambda$. This is called the *wavenumber*.

2.4 Wavenumber Decomposition

Similarly, we can think of the spatial distribution as varying Mathematically, it is the 2D spatial transform of the signal

$$P(k_z, k_y, z) = \mathcal{F}_k\left[\tilde{p}(x, y, z)\right] \equiv \iint_{-\infty}^{\infty} \tilde{p}(x, y, z)\, e^{-i(k_x x + k_y y)}\, \mathrm{d}x\, \mathrm{d}y \tag{17}$$

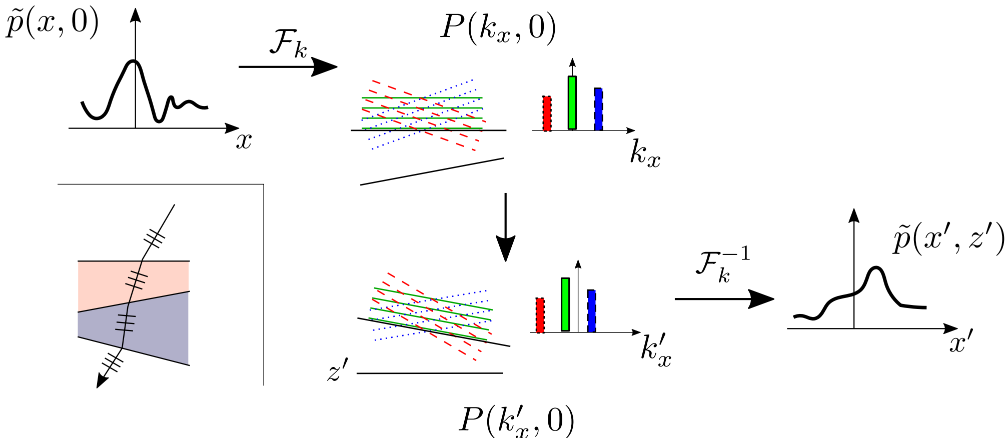


Figure 8: Propagation through layers via rotation of the k-space coordinate system. After Fig. 1 of Ref. 2

So while the signal has one frequency in time, it can also be thought of as a summation of plane waves traveling with different orientations. This field is called the "angular spectrum" or "plane wave decomposition" and is employed for the same reason we look only at one frequency at a time: it makes the math easier. The complex value P is a function of the wavenumber components k_x and k_y. These components define the angle with which that component

2.5 Propagation for Trans-skull Focusing

These plane waves have the nice property that they can be propagated through the layer by a simple phase shift

$$P|_{z+\Delta z} = P|_z e^{ik_z \Delta z}\,, \tag{18}$$

where $k_z^2 = \omega^2/c^2 - k_x^2 - k_y^2$. These plane waves are then rotated at each layered interface to achieve normal incidence (see Fig. 8).

This method also has the advantage that amplitude changes can be included by multiplying by the plane wave transmission coefficient

$$T = \frac{2Z_{\mathrm{ac},2}}{Z_{\mathrm{ac},1} + Z_{\mathrm{ac},2}}\,, \tag{19}$$

where $Z_{\mathrm{ac},i} = \rho_i c_i / \cos\theta_i$ is sometimes called the "acoustic impedance" of the medium (equal to "characteristic" or "specific acoustic impedance" at normal incidence).

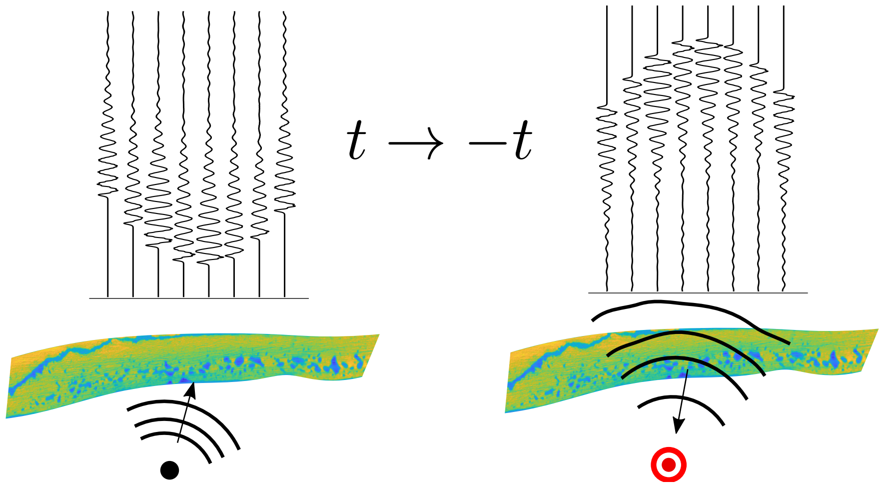


Figure 9: Aberrated signals from a point source are conjugated in time to achieve focusing at that position.

Example 5: Benefits and Limitations

What are the benefits of this approach?
Ans: Fast, can be be performed within minutes.

Limitations?
The method assumes the medium is layered, i.e., that the changes are discrete and no in-plane variation. Also no reflections or mode conversion are accounted for.

3 Time Reversal

Analytical methods are convenient because they are fast and relatively straightforward. However, they often make the assumption of a simplified environment (e.g., homogeneous, discrete layers). How can you go about focusing through something like a skull, which is highly heterogeneous? One way is *time reversal*.[3]

Time reversal relies on *acoustic reciprocity* (see Sec. 4.3), a formal consequence of which is that we can switch the source and receiver positions and get the same result either way. So, if we record the signals due to a point source, we can just turn all our receivers into sources, and the field at the source will just be the point source. This process is illustrated in Fig. 9.

Example 6: Limitations of Time Reversal

Will we get a perfect focal point?
Ans: No—only receiving on finite aperture, so field will be diffracted.

Other problems?
Ans: How do we get a source in there? (We don't; need to use simulation.)

4 For the Interested Reader

4.1 Simplification of Eq. (5)

$$
\begin{aligned}
\Delta t &= \frac{1}{c_1}\frac{z_f}{\sin\theta_1} + \frac{d}{\sin\theta_1}\left(\frac{c_1 - c_2}{c_1 c_2}\right) - \frac{1}{c_1}\frac{z_f}{\sin\theta_2} - \frac{d}{\sin\theta_2}\left(\frac{c_1 - c_2}{c_1 c_2}\right) \\
\left(\frac{c_1 c_2}{c_1 - c_2}\right)\Delta t &= \left(\frac{c_1 c_2}{c_1 - c_2}\right)\frac{1}{c_1}\frac{z_f}{\sin\theta_1} + \frac{d}{\sin\theta_1} - \left(\frac{c_1 c_2}{c_1 - c_2}\right)\frac{1}{c_1}\frac{z_f}{\sin\theta_2} - \frac{d}{\sin\theta_2} \\
&= \left(\frac{c_2}{c_1 - c_2}\right)\frac{z_f}{\sin\theta_1} + \frac{d}{\sin\theta_1} - \left(\frac{c_2}{c_1 - c_2}\right)\frac{z_f}{\sin\theta_2} - \frac{d}{\sin\theta_2} \\
&= \frac{1}{\sin\theta_1}\left[\left(\frac{c_2}{c_1 - c_2}\right) z_f + d\right] - \frac{1}{\sin\theta_2}\left[\left(\frac{c_2}{c_1 - c_2}\right) z_f + d\right] \\
c_1 \Delta t &= \left[z_f + d\left(\frac{c_1}{c_2} - 1\right)\right]\left(\frac{1}{\sin\theta_1} - \frac{1}{\sin\theta_2}\right) \\
\Delta t &= \left[\frac{z_f}{c_1} + d\left(\frac{1}{c_2} - \frac{1}{c_1}\right)\right]\left(\frac{1}{\sin\theta_1} - \frac{1}{\sin\theta_2}\right) .
\end{aligned}
\tag{20}
$$

4.2 Sign Convention

Since $e^{\pm ix} = \cos x \pm i \sin x$, and since we're taking the real part, the choice of sign is arbitrary, i.e., $\mathrm{Re}[e^{ikx}] = \mathrm{Re}[e^{-ikx}]$. But is ***not*** unimportant. The negative convention is used here, because then (in 1D) $p = p_0 e^{i(kx-\omega t)}$ describes a wave traveling in the positive x-direction, which seems more natural to me (a positive convention means forward waves are proportional to e^{-ikx}).

The choice of sign also dictates the sign of the exponents in the Fourier transform kernels. So for the forward time transform, the kernel is $e^{+i\omega t}$, and for the forward ***spatial*** transform, it's $e^{-i(k_x x + k_y y)}$. Either choice (positive or negative) is fine, but make it intentionally and stick with it!

4.3 Reciprocity

A sketch of the proof of reciprocity is presented here; see Refs. 4,5 for more details. Consider the identity

$$
\int_S (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \cdot \boldsymbol{n} \, \mathrm{d}S = \int_V \left(\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1\right) \mathrm{d}V \, ,
\tag{21}
$$

where $S = \partial V$. Now let ϕ_1 and ϕ_2 be the velocity potentials due to two point sources in an arbitrary volume V, but with two small "cutouts" of radii ϵ around the two source positions (see Fig. 10).

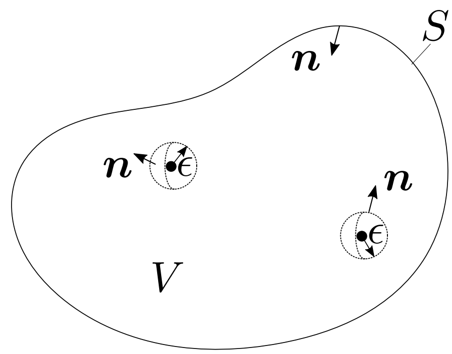


Figure 10: Arbitrary volume with two source regions.

Now within V, sine there are no sources (hence the cutouts) for a harmonic field,† $\nabla^2\phi = -k^2\phi$, so that the right hand side of Eq. (21) vanishes:

$$\int_S (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \cdot \boldsymbol{n} \, \mathrm{d}S = \int_V \left[\phi_1 \left(-k^2 \phi_2 \right) - \phi_2 \left(-k^2 \phi_1 \right) \right] \, \mathrm{d}V = 0 \, . \tag{22}$$

From the definition of the velocity potential

$$p = -i\omega\rho_0\phi \qquad \text{and} \qquad \boldsymbol{u} = \nabla\phi \, , \tag{23}$$

so that the right side of Eq. (21) gives

$$\begin{aligned} \int_S \left[\left(\frac{p_1}{-i\omega\rho_0} \right) (\boldsymbol{u}_2) - \left(\frac{p_2}{-i\omega\rho_0} \right) (\boldsymbol{u}_1) \right] \cdot \boldsymbol{n} \, \mathrm{d}S = 0 \\ \implies \int_S \left(p_1 \boldsymbol{u}_2 - p_2 \boldsymbol{u}_1 \right) \cdot \boldsymbol{n} \, \mathrm{d}S = 0 \, . \end{aligned} \tag{24}$$

Equation (24) indicates that the field at position 2 due to source 1 is identical to the field at position 1 due to source 2. It turns out that the principle is valid for heterogeneous media, but not when losses are included.

References

[1] Pavel B. Rosnitskiy, Petr V. Yuldashev, Oleg A. Sapozhnikov, Leonid R. Gavrilov, and Vera A. Khokhlova. Simulation of nonlinear trans-skull focusing and formation of shocks in brain using a fully populated ultrasound array with aberration correction. *The Journal of the Acoustical Society of America*, 146(3):1786–1798, September 2019.

[2] G. T. Clement and K. Hynynen. A non-invasive method for focusing ultrasound through the human skull. *Physics in Medicine & Biology*, 47(8):1219, 2002.

†As discussed previously, there is no loss of generality here, since more complicated fields can be built up from time harmonic ones.

[3] Mathias Fink, Didier Cassereau, Arnaud Derode, Claire Prada, Philippe Roux, Mickael Tanter, Jean-Louis Thomas, and François Wu. Time-reversed acoustics. *Reports on Progress in Physics*, 63(12):1933, 2000.

[4] Allan D. Pierce. *Acoustics: an Introduction to Its Physical Principles and Applications.* Acoustical Society of America, Melville, NY, 1989.

[5] Lawrence E. Kinsler, Austin R. Frey, Alan B. Coppens, and James V. Sanders. *Fundamentals of Acoustics.* John Wiley and Sons, Inc., New York, 2000.