Georgia Tech Mechanical Engineering Quals Review Book

Unofficial—Use at Own Risk

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To Chris Farley,
whose dedication to physical comedy
should be an example to us all.

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About This Book

"You need a lobotomy, I'll get a saw."
— Bill Waterson, *Calvin and Hobbes*

About the Text

This document is a set of notes created during/for the purpose of reviewing for the Qualifying Examinations in the Acoustics and Applied Mathematics subject areas given by the Woodruff School of Mechanical Engineering at the Georgia Institute of Technology. The topics included were chosen based on information in the Graduate Student Handbook, questions given on past exams, and other insights that came up as we were studying for these tests in the summer and fall of 2017.

My primary intent in keeping these notes was to collect overviews of the topics that were stated as fair game in each subject in one place with a consistent—or at least similar—presentation. The aim in this was to mitigate some of the the confusion I often encounter when switching between different sources caused by differences in notation, terminology, presentation, etc. These notes contain no problems, which are, in my opinion, the only way to prepare for these exams. Hopefully this document can find use as a primer for the topics and techniques that the exams demand.

All material reflects my own understanding of the author, which may or may not bear any resemblance to the Truth.[†] That said, the text is largely amal-

[†]I'm not trying to mislead anyone, but I doubt charlatans ever see themselves that way.

gams and re-presentations of the coverage contained in the sources listed in the bibliography, which are far better resources if a legitimate source is desired. Most are references suggested in the Graduate Student Handbook, with several additions that I found useful (or just happened to have on my desk).

Lastly, it should be made clear that this book has no affiliation with, or endorsement from, Georgia Tech, the Woodruff School, or any faculty or staff involved in administration of the Qualifying Exams. Use this material if you find it useful, but be advised that it is about as authoratative as bathroom stall graffiti.[†]

About the Book

This document was created using the Simple Book LaTeX template[‡] created by Amber Jain,[§] which is licensed under the Open Source Initiative ISC license. Content by the author may be used freely, provided proper attribution of all references (listed beginning on p. 146, and indicated in footnotes throughout) and subject to the below modified MIT license:

[†] For example, I misspelled "authoritative" in that last sentence. If I didn't catch this the first time, who knows what else I missed putting this together.

[‡] `https://github.com/amberj/latex-book-template`

[§] `http://amberj.devio.us/`

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I am indebted to the following people, who were uniquely beneficial resources during preparations for these exams. Most especially, the members of our weekly study group (**bolded)** were sueprlative motivators and teachers as we trekked toward the exam dates together. Though they should not be held liable for any of the material herein, I am thankful for their contributions toward furthering my understanding of math, acoustics, and how the heck to prepare myself for this damn test.

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If you find in these notes any errors, misrepresentations, or misspellings, (or if you think pineapple is an unacceptable pizza topping and want to argue about it), please send me an email.

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Acoustics

> *"[A] deep understanding of acoustical principles is not acquired through superficial efforts."*
>
> — Allan D. Pierce[1]

2.1 Stated Topics

The stated objective of the exam is "to evaluate the student's understanding of the fundamental principles of acoustics. The student must demonstrate the ability to attack problems with a correct approach and show the ability to analyze problems and results with critical judgment in a manner compatible with doctoral level expectations."

While historically the exam has covered topics from both ME 6760 and 6761 (Acoustics I and II), recent years seemed to have focused more on material covered in the first semester course. Thus questions on multipole expansions, non-rectangular enclosures, and general radiation problems are probably less likely on the written exam. But they're not out of bounds, so they're included here (though in a bit less detail).

Finally, since the fall 2017 exam permitted a single page crib sheet, I've included mine in the Acoustics Appendix (p. 139). It didn't prove of much use during that exam, but I thought it was a good exercise to condense these notes to a single page (and to underscore some details I found myself likely to forget). As with everything in this document, use it however you like, but at your own risk.

2.2 Fundamentals

2.2.1 Governing Equations

To derive the wave equation for a fluid, we'll employ three fundamental laws: conservation of mass, conservation of momentum, and an equation of state.[†] We'll then restrict analysis to small perturbations of the pressure, density, and with no mean flow to obtain the linear, lossless wave equation.

Continuity Equation

The continuity equation relates the amount of matter inside some volume V to the rate at which mass is flowing in or out through the surface which bounds the volume. If the fluid has density total ϱ, the total mass inside some arbitrary volume is

$$(\text{Mass in } V) = \int_V \varrho \, \mathrm{d}V \,. \tag{2.2.1}$$

But how does the amount of mass in V change with time? Well, the mass in V can change in two ways:

1. **When fluid enters or leaves V through its surface ∂V** Suppose the fluid is flowing *out* at some rate $\boldsymbol{q} = \varrho \boldsymbol{v}$.[‡] Then the amount of mass that flows out through the surface is per unit time

$$\begin{aligned} \big(\text{Change of Mass in } V\big) &= -\big(\text{Mass Flowing Out of } V\big) \\ \implies \dot{m}_{\text{outflow}} &= -\int_{\partial V} \boldsymbol{q} \cdot \boldsymbol{e}_n \, \mathrm{d}S \,. \end{aligned} \tag{2.2.2}$$

2. **When fluid is injected somewhere in V** In principle, there's no reason why we couldn't have a mass injection source in the volume. If the source is adding mass within V at some rate Σ

$$\begin{aligned} \big(\text{Change of Mass in } V\big) &= (\text{Mass Created in } V) \\ \implies \dot{m}_{\text{injection}} &= \Sigma \,. \end{aligned} \tag{2.2.3}$$

[†] This limits our anaysis to lossless propagation. If we wanted to include, e.g., viscous or thermal losses, we'd need an energy equation to describe these mechanisms.

[‡] We choose to measure the flow out rather than in because the positive unit normal vector of a surface is conventionally oriented outward.

Typically, we don't have any injection sources, so we'll set $\Sigma = 0$. Then, since Eq. (2.2.1) gives the total mass in V, we can differentiate it to with respect to time to get the rate of change of the mass in V. Then, we set that equal to Eq. (2.2.2), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_V \varrho \,\mathrm{d}V &= -\int_{\partial V} \boldsymbol{q}\cdot\boldsymbol{e}_n \,\mathrm{d}S \\ \int_V \frac{\partial \varrho}{\partial t}\,\mathrm{d}V &= -\int_{\partial V} \boldsymbol{q}\cdot\boldsymbol{e}_n \,\mathrm{d}S \\ \int_V \frac{\partial \varrho}{\partial t}\,\mathrm{d}V &= -\int_V \nabla\cdot\boldsymbol{q}\,\mathrm{d}V \\ \implies \int_V \left(\frac{\partial \varrho}{\partial t} + \nabla\cdot\boldsymbol{q}\right)\mathrm{d}V &= 0, \end{aligned} \tag{2.2.4}$$

where we've used the divergence theorem.[†] But, we haven't said anything about V! The only way that Eq. (2.2.4) will vanish for *any* V we choose is if its integrand is uniformly 0. Then, since the flux $\boldsymbol{q} = \varrho\boldsymbol{v}$, we arrive at the continuity equation:

$$\boxed{\frac{\partial \varrho}{\partial t} + \nabla\cdot(\varrho\boldsymbol{v}) = 0\,.} \tag{2.2.5}$$

Momentum Equation

We saw in Sec. 2.2.1 that the change in mass in the control volume V is equal to the flow of mass through the boundary plus any mass created within the volume. We can perform a similar analysis for the momentum. That is:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\,(\text{Momentum in } V) = \\ &\qquad \big(\text{Momentum Flow } \textit{In} \text{ through } \partial V\big) \;+\; (\text{Net Force on } V)\,. \end{aligned} \tag{2.2.6}$$

The momentum in V is simply

$$\begin{aligned} (\text{Momentum in } V) &= \int_V \varrho\boldsymbol{v}\,\mathrm{d}V \\ \implies \frac{\mathrm{d}}{\mathrm{d}t}\,(\text{Momentum in } V) &= \int_V \frac{\partial}{\partial t}\,(\varrho\boldsymbol{v})\,\mathrm{d}V \end{aligned} \tag{2.2.7}$$

[†]See Sec. 3.2.3.

Now the momentum flow inward through the boundary is[†]

$$\begin{aligned}\left(\text{Momentum Flow } \textit{In} \text{ through } \partial V\right) &= -\int_{\partial V} (\rho \boldsymbol{v})\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s} \\ &= -\int_V \nabla \cdot [(\rho \boldsymbol{v})\boldsymbol{v}] \, \mathrm{d}V\,, \end{aligned} \tag{2.2.8}$$

where we've used the divergence theorem. Finally, assuming that the fluid is inviscid (i.e., there are no shear forces), we have that the only force exerted on V is due to the pressure:[‡]

$$\begin{aligned}(\text{Net Force on } V) &= \int_{\partial V} -P \, \mathrm{d}S \\ &= -\int_V \nabla \cdot P \, \mathrm{d}V\,, \end{aligned} \tag{2.2.9}$$

where we've again used the divergence theorem. Now, substituting Eqs. (2.2.7) to (2.2.9) into Eq. (2.2.6)

$$\begin{aligned}\left[\int_V \frac{\partial}{\partial t}(\varrho \boldsymbol{v}) \, \mathrm{d}V\right] = \left[-\int_V \nabla \cdot [(\varrho \boldsymbol{v})\boldsymbol{v}] \, \mathrm{d}V\right] + \left[-\int_V \nabla \cdot P \, \mathrm{d}V\right] \\ \implies \int_V \frac{\partial}{\partial t}(\varrho \boldsymbol{v}) + \nabla \cdot [(\varrho \boldsymbol{v})\boldsymbol{v}] + \nabla P \, \mathrm{d}V = 0\,. \end{aligned} \tag{2.2.10}$$

And again, we've said nothing about the nature of V, meaning that the integrand must vanish for *any* V, and therefore must be 0:

$$\frac{\partial}{\partial t}(\varrho \boldsymbol{v}) + \nabla \cdot [(\rho \boldsymbol{v})\boldsymbol{v}] + \nabla P = 0\,. \tag{2.2.11}$$

Now expanding the terms in Eq. (2.2.11),

$$\begin{aligned}&\frac{\partial \varrho}{\partial t}\boldsymbol{v} + \rho \frac{\partial \boldsymbol{v}}{\partial t} + \nabla \cdot (\varrho \boldsymbol{v})\boldsymbol{v} + (\rho \boldsymbol{v})\nabla \cdot \boldsymbol{v} + \nabla P = 0 \\ \implies &\underbrace{\left[\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \boldsymbol{v})\right]}_{=0} \boldsymbol{v} + \varrho \frac{\partial \boldsymbol{v}}{\partial t} + (\varrho \boldsymbol{v})\nabla \cdot \boldsymbol{v} + \nabla P = 0 \end{aligned} \tag{2.2.12}$$

[†]As with the mass flow, the positive outward normal means that considering *inflow* incurs a negative sign.

[‡]The negative sign is due to the fact that a positive pressure (which causes a compression) will be inward, while the normal to the surface is outward. Then $\mathrm{d}\boldsymbol{F} = -P\boldsymbol{e}_n \, \mathrm{d}S$

(notice the bracketed term in Eq. (2.2.12) is the left hand side of the continuity relation Eq. (2.2.5), which vanishes). So then we have;

$$\underbrace{\varrho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\nabla\cdot\boldsymbol{v}\right)}_{=\mathrm{D}\boldsymbol{v}/\mathrm{D}t} + \nabla P = 0$$

$$\boxed{\varrho\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + \nabla P = 0\,.} \tag{2.2.13}$$

Another derivation of Eq. (2.2.13) is given in Sec. A.1.

Equation of State

Generally, we will need to know how the density, entropy, and pressure of the medium are related. Since a passing sound wave induces changes in pressure, we'll need to know how this affects the other properties of the medium.

Pressure-Density Relations

For a perfect gas, the equation of state is the familiar ideal gas law

$$\begin{aligned} PV &= NR_gT \\ \implies P &= \rho R_g T\,, \end{aligned} \tag{2.2.14}$$

where the gas constant R_g depends on the type of gas. If the gas at some pressure and density P_0 and ρ_0 is compressed slowly (quasi-statically), so that the gas has time to equilibrate with its surroundings, then the gas will stay at the same temperature. In this case, we can write from Eq. (2.2.14)

$$\begin{aligned} T_1 &= T_2 \\ \left(\frac{P_0}{\rho_0 R_g}\right) &= \left(\frac{P}{\rho R_g}\right) \\ \frac{P}{P_0} &= \frac{\rho}{\rho_0} \quad \text{(isothermal)}\,. \end{aligned} \tag{2.2.15}$$

Similarly, if the compression happens so quickly that there's no time for heat to flow from the gas, we'll have adiabatic compression[†]

$$\frac{P}{P_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma} \quad \text{(adiabatic)}\,. \tag{2.2.16}$$

[†]If you care where this relationship comes from, see Sec. A.2.

At acoustic pressures and frequencies, we will almost always have an adiabatic process.[2]

But it's generally hard to write the state equation that accounts for behavior other than a perfect gas. Instead, we'll assume that the compression is adiabatic,[†] such that the entropy S is constant. Then $P = P(\varrho)$ the pressure as a function of the excess density $\Delta\varrho$ can be written as a Taylor expansion

$$P = P_0 + \left(\frac{\partial P}{\partial \varrho}\right)_{\rho_0} \Delta\varrho + \frac{1}{2}\left(\frac{\partial P}{\partial \varrho}\right)^2_{\rho_0} (\Delta\varrho)^2 + \mathcal{O}\left[(\Delta\varrho)^3\right] . \tag{2.2.17}$$

We note that the adiabatic bulk modulus of the fluid is

$$K \equiv \rho_0 \left(\frac{\partial P}{\partial \varrho}\right)_S = -V\left(\frac{\partial P}{\partial V}\right)_S , \tag{2.2.18}$$

where the S subscript indicates the derivative is to be taken at a constant entropy. Then if we keep only linear terms, we have from Eqs. (2.2.17) and (2.2.18), we have

$$\begin{aligned} P - P_0 &\simeq K\left(\frac{\varrho - \rho_0}{\rho_0}\right) \\ p &\simeq Ks, \end{aligned} \tag{2.2.19}$$

where $s = (\varrho - \rho_0)/\rho_0 = \rho/\rho_0$ is called the condensation. If we were to model the fluid as a perfect gas, we could use the adiabatic relationship Eq. (2.2.16). Then keeping terms in the expansion Eq. (2.2.17), we could show that

$$\begin{aligned} p &= P_0\left[\gamma s + \frac{1}{2}\gamma(\gamma-1)s^2 + \ldots\right] \\ &= \gamma P_0\, s + \frac{\gamma P_0}{2}(\gamma-1)s^2 \\ &= K\, s + \frac{B}{2A}\, K\, s^2, \end{aligned} \tag{2.2.20}$$

where $B/2A = (\gamma - 1)/2$ is the parameter of nonlinearity. Typically (and most importantly, on the qualifying exam) we're concerned with linear acoustics, where $s \ll 1$, and thus only the first term in Eq. (2.2.20) is retained.

[†]Really we mean isentropic, since the process is assumed to be reversible as well.

Speed of Sound

The thermodynamic speed of sound is defined to be

$$c^2 \equiv \left(\frac{\partial P}{\partial \varrho}\right)_S . \tag{2.2.21}$$

Note that in general, the sound speed is a function. However, usually we take it to be constant, or effectively, that the range of pressures is small enough that relation $P(\varrho)$ is effectively linear (i.e., its slope is constant) over the range. If we use the definition of the bulk modulus [Eq. (2.2.18)], and evaluate near the ambient density ρ_0, we can write

$$c_0^2 = \frac{\rho_0}{\rho_0}\left(\frac{\partial P}{\partial \varrho}\right)_S = \frac{1}{\rho_0} K . \tag{2.2.22}$$

Here c_0 indicates that we have taken the sound speed to be a constant, namely the *small signal* speed of sound.

Consider now propagation in a gas Since acoustic compression and rarefaction is almost always adiabatic, we will have PV^γ = constant, or

$$\frac{P}{\varrho^\gamma} = \frac{P_0}{\varrho_0^\gamma} . \tag{2.2.23}$$

Therefore,

$$\begin{aligned} c^2 = \left(\frac{\partial P}{\partial \varrho}\right)_S &= \frac{\partial}{\partial \varrho}\left[\varrho^\gamma \frac{P_0}{\varrho_0^\gamma}\right] \\ &= \gamma \varrho^{\gamma-1} \frac{P_0}{\varrho_0^\gamma} = \frac{\gamma P_0}{\varrho_0}\left(\frac{\varrho^{\gamma-1}}{\varrho_0^{\gamma-1}}\right) . \end{aligned} \tag{2.2.24}$$

In the case where $s \ll 1$, we have that the last term in Eq. (2.2.24) is

$$\frac{\varrho^{\gamma-1}}{\varrho_0^{\gamma-1}} = \left[\frac{(\rho_0+\rho_2)}{(\rho_0+\rho_1)}\right]^{\gamma-1} = \left(\frac{1+s}{1+s_0}\right)^{\gamma-1} \simeq 1 . \tag{2.2.25}$$

We've chosen our reference density to be the ambient density, such that $s_0 = 0$. Thus for adiabatic propagation,[†]

$$c_0 = \sqrt{\frac{\gamma P_0}{\rho_0}} . \tag{2.2.26}$$

[†]Since many references seems to mention it,[1,3–5] note that Newton derived the speed of sound as $c^2 = P_0/\rho_0$ since he assumed isothermal compression (i.e., that Boyle's law PV = constant held). He could hardly be blamed, since thermodynamics wasn't even a field yet.

2.2.2 Wave Equation Derivation

Now we'll derive the wave familiar wave equation. The strategy is to suppose that acoustic perturbations of the total pressure and density are small compared to the ambient[†] values, and retain only first-order terms. We write

$$P = p_0 + p \qquad (\|p/p_0\| \ll 1) \tag{2.2.27}$$
$$\varrho = \rho_0 + \rho \qquad (\|\rho/\rho_0\| \ll 1) \tag{2.2.28}$$
$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{u} = \boldsymbol{u} \qquad (\|\boldsymbol{u}/c_0\| \ll 1) \tag{2.2.29}$$

If we substitute these expansions into the continuity equation [Eq. (2.2.5)], we get

$$\begin{aligned} \frac{\partial}{\partial t}(\rho_0 + \rho) + \nabla \cdot [(\rho_0 + \rho)\,\boldsymbol{u}] &= 0 \\ \underline{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{u} + \rho \boldsymbol{u}) &= 0\,. \end{aligned} \tag{2.2.30}$$

Terms that vanish will be underlined. Expanding the divergence in Eq. (2.2.30) gives

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{u} + \nabla \cdot (\rho \boldsymbol{u}) &= 0 \\ \frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{u} + \underline{\underline{\rho \nabla \cdot \boldsymbol{u}}} + \underline{\underline{\boldsymbol{u} \cdot \nabla \rho}} &= 0\,, \end{aligned} \tag{2.2.31}$$

where second-order terms are underlined twice. Retaining only first-order terms, we arrive at the linearized continuity equation

$$\boxed{\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{u} = 0\,.} \tag{2.2.32}$$

Now we substitute our perturbed variables into the momentum equation [Eq. (2.2.13)]:

$$\begin{aligned} \varrho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + \nabla P &= 0 \\ (\rho_0 + \rho)\left[\frac{\partial}{\partial t}(\boldsymbol{v}_0 + \boldsymbol{u}) + \underline{\boldsymbol{v}_0 \nabla \cdot (\boldsymbol{v}_0 + \boldsymbol{u})}\right] + \nabla (p_0 + p) &= 0 \\ \underline{(\rho_0 + \rho)\frac{\partial \boldsymbol{v}_0}{\partial t}} + \rho_0 \frac{\partial \boldsymbol{u}}{\partial t} + \underline{\underline{\rho \frac{\partial \boldsymbol{u}}{\partial t}}} + \underline{\nabla p_0} + \nabla p &= 0\,. \end{aligned} \tag{2.2.33}$$

[†]We also assume that the medium is homogeneous, i.e., that its material properties do not vary in space (or time).

Discarding second-order terms and those that vanish gives the linearized momentum (or Euler's) equation

$$\boxed{\rho_0 \frac{\partial \boldsymbol{u}}{\partial t} + \nabla p = 0\,.} \tag{2.2.34}$$

Finally, our small signal sound speed is found by retaining only the first term of Eq. (2.2.20), such that

$$\begin{aligned} p &= K\, s \\ &= \left(\rho_0 c_0^2\right)\left(\frac{\varrho - \rho_0}{\rho_0}\right) \\ &= \rho c_0^2, \end{aligned} \tag{2.2.35}$$

so that

$$\boxed{c_0^2 = \frac{p}{\rho}} \tag{2.2.36}$$

Now, to obtain the wave equation, first differentiate Eq. (2.2.36) with respect to time

$$\begin{aligned} &\rho c_0^2 = p \\ \implies &\frac{\partial \rho}{\partial t} = c_0^{-2} \frac{\partial p}{\partial t}\,. \end{aligned} \tag{2.2.37}$$

Then substitute into Eq. (2.2.32)

$$\begin{aligned} &\left(\frac{1}{c_0^2}\frac{\partial p}{\partial t}\right) + \rho_0 \nabla \cdot \boldsymbol{u} = 0 \\ \implies &\frac{\partial p}{\partial t} + \rho_0 c_0^2 \nabla \cdot \boldsymbol{u} = 0\,. \end{aligned} \tag{2.2.38}$$

Differentiating Eq. (2.2.38) with respect to time gives

$$\frac{\partial^2 p}{\partial t^2} + \rho_0 c_0^2 \nabla \cdot \frac{\partial \boldsymbol{u}}{\partial t} = 0\,. \tag{2.2.39}$$

Taking the divergence of Eq. (2.2.34) and multiplying by c_0^2 gives

$$\rho_0 c_0^2 \nabla \cdot \frac{\partial \boldsymbol{u}}{\partial t} + c_0^2 \nabla^2 p = 0\,. \tag{2.2.40}$$

Finally, subtract Eq. (2.2.39) from Eq. (2.2.40) to get the linear wave equation:

$$\boxed{\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = 0\,.} \tag{2.2.41}$$

2.2.3 Modified Wave Equations

Past exams have asked to derive a modified wave equation, involving a modification to one of the assumptions. Usually the governing equations are given, but they are derived below for completeness.

Mean Flow

Suppose that our small signal and homogeneous medium assumptions hold, but now the medium is traveling with a constant mean flow such that $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{U}_0$. In this case the continuity equation [Eq. (2.2.5)] gives

$$\begin{aligned} \frac{\partial}{\partial t}(\rho_0 + \rho) + \nabla \cdot [(\rho_0 + \rho)(\boldsymbol{U}_0 + \boldsymbol{u})] &= 0 \\ \underline{\frac{\partial \rho_0}{\partial t}} + \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho_0 \boldsymbol{u} + \rho \boldsymbol{u} + \rho_0 \boldsymbol{U}_0 + \rho \boldsymbol{U}_0) &= 0 \\ \frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{u} + \underline{\underline{\nabla \rho \cdot \boldsymbol{u}}} + \underline{\underline{\rho \nabla \cdot \boldsymbol{u}}} + \underline{\rho_0 \nabla \cdot \boldsymbol{U}_0} + \nabla \rho \cdot \boldsymbol{U}_0 &= 0\,, \end{aligned} \tag{2.2.42}$$

where we've again underlined vanishing terms and underlined twice second order terms to be discarded. Then since we still have our linearized sound speed,

$$c_0^2 = p/\rho \quad \Longrightarrow \quad \nabla \rho = \nabla p / c_0^2\,, \tag{2.2.43}$$

$$\Longrightarrow \quad \frac{\partial \rho}{\partial t} = \frac{1}{c_0^2}\left(\frac{\partial p}{\partial t}\right) \tag{2.2.44}$$

we have that Eq. (2.2.42) becomes

$$\frac{\partial p}{\partial t} + \boldsymbol{U}_0 \cdot \nabla p + \rho_0 c_0^2 \nabla \cdot \boldsymbol{u} = 0\,. \tag{2.2.45}$$

The momentum equation [Eq. (2.2.13)] is modified too, since the full material derivative must be included

$$\begin{aligned} (\rho_0 + \rho)\left[\frac{\partial}{\partial t}(\boldsymbol{U}_0 + \boldsymbol{u}) + \boldsymbol{U}_0 \cdot \nabla (\boldsymbol{U}_0 + \boldsymbol{u})\right] + \nabla(p_0 + p) &= 0 \\ (\rho_0 + \rho)\left[\underline{\frac{\partial \boldsymbol{U}_0}{\partial t}} + \frac{\partial \boldsymbol{u}}{\partial t} + \underline{(\boldsymbol{U}_0 \cdot \nabla)\boldsymbol{U}_0} + (\boldsymbol{U}_0 \cdot \nabla)\boldsymbol{u}\right] + \underline{\nabla p_0} + \nabla p &= 0\,. \end{aligned} \tag{2.2.46}$$

Note the operator $(\boldsymbol{a}\cdot\nabla)\boldsymbol{b}$ is just $a_x\partial\boldsymbol{b}/\partial x + a_y\partial\boldsymbol{b}/\partial y + a_z\partial\boldsymbol{b}/\partial z$. If we again keep only first-order terms, the momentum equation becomes

$$\rho_0\left[\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{U}_0\cdot\nabla)\,\boldsymbol{u}\right] + \underline{\underline{\rho\frac{\partial\boldsymbol{u}}{\partial t}}} + \underline{\underline{\rho\,(\boldsymbol{U}_0\cdot\nabla)\,\boldsymbol{u}}} + \nabla p = 0$$

$$\rho_0\left[\frac{\partial\boldsymbol{u}}{\partial t} + (\boldsymbol{U}_0\cdot\nabla)\,\boldsymbol{u}\right] + \nabla p = 0\,. \tag{2.2.47}$$

To find the wave equation, take the time derivative of Eq. (2.2.45)

$$\frac{\partial^2 p}{\partial t^2} + \boldsymbol{U}_0\cdot\nabla p_t + \rho_0 c_0^2\nabla\cdot\boldsymbol{u}_t = 0\,, \tag{2.2.48}$$

and take c_0^2 times the divergence of Eq. (2.2.47) to get

$$\rho_0 c_0^2\,\nabla\cdot\boldsymbol{u}_t + \rho_0 c_0^2\,\nabla\cdot(\boldsymbol{U}_0\cdot\nabla)\,\boldsymbol{u} + c_0^2\nabla^2 p = 0\,. \tag{2.2.49}$$

Now subtract Eq. (2.2.49) from Eq. (2.2.48) to eliminate the $\nabla\cdot\boldsymbol{u}_t$ terms

$$\frac{\partial^2 p}{\partial t^2} + \boldsymbol{U}_0\cdot\nabla p_t - c_0^2\nabla^2 p - \left[\rho_0 c_0^2\,\nabla\cdot(\boldsymbol{U}_0\cdot\nabla)\,\boldsymbol{u}\right] = 0\,, \tag{2.2.50}$$

Now we need to handle the bracketed term in Eq. (2.2.50). Take the gradient of the continuity equation [Eq. (2.2.45)] and dot it with $\boldsymbol{U}_0$ to get

$$\boldsymbol{U}_0\cdot\nabla p_t + U_0^2\cdot\nabla^2 p + = -\rho_0 c_0^2\nabla\cdot(\boldsymbol{U}_0\cdot\nabla)\boldsymbol{u}\,, \tag{2.2.51}$$

where $U_0 = \|\boldsymbol{U}_0\|$ But the right hand side of Eq. (2.2.51) is precisely the bracketed term in Eq. (2.2.50)! Substituting it in, we obtain the wave equation for mean flow

$$\frac{\partial^2 p}{\partial t^2} + 2\boldsymbol{U}_0\cdot\nabla p_t - c_0^2\,\nabla^2 p + U_0^2\,\nabla^2 p = 0\,. \tag{2.2.52}$$

Notice that if we define a coordinate system that moves with the fluid

$$\boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{U}_0 t\,,\quad \tau = t\,, \tag{2.2.53}$$

such that

$$\nabla p = \nabla' p\,,\qquad \text{and} \tag{2.2.54}$$

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial \tau} - U_0\nabla' p\,, \tag{2.2.55}$$

we recover the familiar result

$$\frac{\partial^2 p}{\partial \tau^2} - c_0^2\nabla'^2 p = 0\,. \tag{2.2.56}$$

2.2.4 Solutions to the Wave Equation

If we consider the one-dimensional wave equation

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0\,, \tag{2.2.57}$$

we can factor the operator as

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)p = 0\,. \tag{2.2.58}$$

Suppose then we write

$$\xi \equiv x - ct \tag{2.2.59}$$

$$\eta \equiv x + ct\,. \tag{2.2.60}$$

Then $\partial/\partial x = \partial/\partial\xi + \partial/\partial\eta$ and $\partial/\partial t = -c(\partial/\partial\xi - \partial/\partial\eta)$. Then, Eq. (2.2.58) can be written as

$$\begin{aligned}\left[\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta} - \left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right)\right]\left[\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta} + \left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right)\right]p &= 0\\ \left(2\frac{\partial}{\partial\eta}\right)\left(2\frac{\partial}{\partial\xi}\right)p &= 0\\ \frac{\partial^2 p}{\partial\eta\partial\xi} &= 0\,.\end{aligned} \tag{2.2.61}$$

So clearly any function that is a function of just ξ or η will satisfy the wave equation. That is, solutions have the form

$$p = f(x - ct) + g(x + ct)\,. \tag{2.2.62}$$

Sinusoidal Waves While any solution of Eq. (2.2.62) satisfies the wave equation, it is typically convenient to work with wave components of a single frequency. While this may seem restrictive, note that we can break any signal up into sinusoidal components (see Sec. 2.3.1). Provided our system is linear, we can perform all operations on each frequency component, sum the results, and take the inverse transform to obtain the time-series result.

If we have just a single angular frequency $\omega = 2\pi f$, then we can write

$$\xi = x - ct = c\,(x/c - t) = \frac{c}{\omega}\left(\frac{\omega}{c}x - \omega t\right) \propto kx - \omega t\,. \tag{2.2.63}$$

Thus a function of $kx - \omega t$ is also a function of ξ up to a change of variables. That is to say, Eq. (2.2.61) is satisfied, and therefore a function $f(kx - \omega t)$ is a solution of the wave equation.[†] Now the sinusoidal function has the form

$$p(x,t) = A\sin(kx - \omega t) + B\cos(kx - \omega t)\,. \tag{2.2.64}$$

To verify, notice that

$$\begin{aligned}\frac{\partial^2 p}{\partial t^2} &= \frac{\partial}{\partial t}\left[-\omega A\cos(kx-\omega t) + \omega B\sin(kx-\omega t)\right]\\ &= -\omega^2 A\sin(kx-\omega t) - \omega^2 B\cos(kx-\omega t)\\ &= -\omega^2\left[A\sin(kx-\omega t) + B\cos(kx-\omega t)\right]\\ &= -\omega^2 p(x,t)\,,\end{aligned} \tag{2.2.65}$$

and

$$\begin{aligned}\frac{\partial^2 p}{\partial x^2} &= \frac{\partial}{\partial x}\left[kA\cos(kx-\omega t) - kB\sin(kx-\omega t)\right]\\ &= -k^2 A\sin(kx-\omega t) - k^2 B\cos(kx-\omega t)\\ &= -k^2\left[A\sin(kx-\omega t) + B\cos(kx-\omega t)\right]\\ &= -k^2 p(x,t)\,,\end{aligned} \tag{2.2.66}$$

and therefore Eq. (2.2.57) becomes

$$\begin{aligned}\left[-k^2 p(x,t)\right] - \frac{1}{c^2}\left[-\omega^2 p(x,t)\right] &\stackrel{?}{=} 0\\ \left(-k^2 + \frac{\omega^2}{c^2}\right)p(x,t) &\stackrel{\checkmark}{=} 0\,.\end{aligned} \tag{2.2.67}$$

However, it was a bit of a slog to keep track of the sines and cosines as we took the derivatives. To make things a bit more convenient, first recall Euler's formula, which says that

$$e^{ix} = \cos x + i\sin x\,. \tag{2.2.68}$$

[†]Note that if $\zeta = ax$ for a scalar a,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[f(\zeta)\right] &= \frac{\mathrm{d}f}{\mathrm{d}\zeta}\cdot\frac{\mathrm{d}\zeta}{\mathrm{d}x}\\ &= f'(\zeta)\cdot(a) = af'(ax)\,.\end{aligned}$$

So if $f(x)$ is a solution of

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = 0\,,$$

then so too is $f(ax)$.

Then we can write, for example,

$$\cos x = \mathrm{Re}\, e^{ix}, \text{ and } \sin x = \mathrm{Re}\, -i\, e^{ix}. \tag{2.2.69}$$

Now, Eq. (2.2.64) can be written as

$$\begin{aligned} p(x,t) &= \mathrm{Re}\left[-iAe^{i(kx-\omega t)}\right] + \mathrm{Re}\left[Be^{i(kx-\omega t)}\right] \\ &= \mathrm{Re}\left[(B-iA)e^{i(kx-\omega t)}\right]. \end{aligned} \tag{2.2.70}$$

We'll define the complex amplitude $C = B - iA$, and perform the same procedure for the sinusoid with argument $-i(kx + \omega t)$. We see that single-frequency solutions to the 1D wave equation will have the form

$$p(x,t) = C_1 e^{i(kx-\omega t)} + C_2 e^{-i(kx+\omega t)} \tag{2.2.71}$$

But wait, we've just written the pressure as a complex number. Of course measured quantities must be real. So we'll adopt the convention that *take real operator as implicit.* That is, we will write the complex pressure Eq. (2.2.71) with the understanding that the measured pressure will be the real part.[†]

The main advantage of complex numbers is that derivatives of exponential functions are trivial. For example, if $p = C_1 \exp i(kx - \omega t)$, then

$$\begin{aligned} \frac{\partial p}{\partial t} &= \frac{\partial}{\partial t}\left[C_1 e^{i(kx-\omega t)}\right] \\ &= -i\omega p \end{aligned} \tag{2.2.72}$$

Plane Waves

Plane waves represent a solution to the harmonic wave equation in 1D, provided our x direction is aligned with the direction of propagation. These waves can be written

$$p = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}, \tag{2.2.73}$$

which represent forward- and backward travelling waves. The associated particle velocity can be found from the momentum equation [Eq. (2.2.34)] to be

$$u = \frac{A}{\rho_0 c_0} e^{i(kx-\omega t)} - \frac{B}{\rho_0 c_0} e^{-i(kx+\omega t)}. \tag{2.2.74}$$

[†]Note that a negative time convention is used in these notes, i.e., $p \propto e^{-i\omega t}$. Results can be converted to a positive time convention (as is used in my primary source, Ref. 4,6) by replacing $i \to -j$.

Spherical Waves

The other fundamental 1D solution to the wave equation is for spherically symmetric geometries. If we write the wave equation in spherical coordinates, and assume that we have no dependence on θ or ϕ, we have

$$\frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = 0\,. \tag{2.2.75}$$

But notice we could rearrange Eq. (2.2.75)

$$\begin{aligned} \underbrace{r\frac{\partial^2 p}{\partial r^2} + 2\frac{\partial p}{\partial r}}_{\partial^2/\partial r^2(rp)} - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}(rp) &= 0 \\ \frac{\partial^2}{\partial r^2}(rp) - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}(rp) &= 0\,. \end{aligned} \tag{2.2.76}$$

Thus we conclude that rp is a solution to the wave equation and thus has the form of the general solution Eq. (2.2.62):

$$\begin{aligned} rp &= f(r+ct) + g(r-ct) \\ p &= \frac{f(r+ct)}{r} + \frac{f(r-ct)}{r}\,. \end{aligned} \tag{2.2.77}$$

For a harmonic solution then,

$$p = \frac{A}{r}\,e^{i(kr-\omega t)} + \frac{B}{r}\,e^{-i(kr+\omega t)}\,. \tag{2.2.78}$$

Now the velocity is a bit less trivial than for the plane wave case. Since $\nabla \to \partial/\partial r$, we have

$$\begin{aligned} -i\omega\rho_0 u &= \frac{\partial}{\partial r}\left[\frac{A}{r}\,e^{i(kr-\omega t)} + \frac{B}{r}\,e^{-i(kr+\omega t)}\right] \\ &= (ik - 1/r)\,\frac{A}{r}\,e^{i(kr-\omega t)} + (-ik + 1/r)\,\frac{B}{r}\,e^{-i(kr+\omega t)} \\ u &= \frac{1}{\rho_0 c_0}\left[\Big(1 - 1/ikr\Big)p_+ - \Big(1 - 1/ikr\Big)p_-\right]\,. \end{aligned} \tag{2.2.79}$$

2.2.5 Acoustic Quantities

Specific Acoustic Impedance

Most generally, the impedance seen by a wave is the ratio of the acoustic pressure to the acoustic particle velocity

$$Z \equiv \frac{p}{u}\,. \tag{2.2.80}$$

If we consider a progressive plane wave, travelling in the positive x direction, then we have

$$p = Ae^{-i(\omega t - kx)}\,. \tag{2.2.81}$$

For a linear medium with no mean flow, the momentum equation [Eq. (2.2.13)] becomes

$$\rho_0 \frac{\partial \boldsymbol{u}}{\partial t} + \nabla p = 0\,. \tag{2.2.82}$$

Substituting Eq. (2.2.81) into Eq. (2.2.82) gives

$$\begin{aligned} \rho_0(-i\omega)\boldsymbol{u} + (ik)Ae^{-i(\omega t - kx)}\boldsymbol{e}_x &= p \\ \implies \rho_0\omega\boldsymbol{u} &= kp\boldsymbol{e}_x\,. \end{aligned} \tag{2.2.83}$$

Clearly then the particle velocity $\boldsymbol{u} = u\boldsymbol{e}_x$ is only in the x-direction (the direction of propagation), and so

$$\begin{aligned} \rho_0\omega u &= \left(\frac{\omega}{c_0}\right) p \\ \implies u &= \frac{1}{\rho_0 c_0} p\,. \end{aligned} \tag{2.2.84}$$

So for a progressive plane wave,

$$Z = \rho_0 c_0. \tag{2.2.85}$$

This quantity, which is a property of the medium, is the called the *specific acoustic impedance* or *characteristic impedance.*

Sound Pressure Level

The sound pressure level (SPL) is a level that is used to account for the enormous range of pressure values that typical sound comprises. It is defined by

$$L_p = 10\log_{10}\left(\frac{p_{\mathrm{rms}}}{p_{\mathrm{ref}}}\right)^2. \tag{2.2.86}$$

and is measured in decibels. The RMS pressure is calculated from

$$p_{\mathrm{rms}}^2 \equiv \frac{1}{T}\int_0^T p^2\,\mathrm{d}t\,, \tag{2.2.87}$$

where T is some appropriate time interval. But what should this T be? If our signal is harmonic, our best option is simply the period of the wave, since in this case

$$p_{\text{rms}}^2 = \frac{1}{(2\pi/\omega)} \int_0^{2\pi/\omega} (\cos \omega t)^2 \, \mathrm{d}t = \frac{1}{2} \,. \tag{2.2.88}$$

Note that for two sources, in general, we cannot simply add RMS pressures. We must first sum the individual pressures and then find the RMS of the *total* pressure:

$$\begin{aligned} p_{\text{tot}} &= \sum_n p_n \\ \implies p_{\text{tot,rms}}^2 &= \frac{1}{T} \int_0^T \left(\sum_n p_n \right)^2 \mathrm{d}t \\ &= \sum_n p_{n,\text{rms}}^2 + (\text{cross terms}) \,. \end{aligned} \tag{2.2.89}$$

In general, we do need to account for these cross terms. However, if the sources are *incoherent* (that is there is no constant phase difference between them) then we can say that on average, they are 0. In this case, we may just sum the RMS pressures

$$p_{\text{tot,rms}}^2 = \sum_n p_{n,\text{rms}}^2 \quad (\text{incoherent sources}) \,. \tag{2.2.90}$$

If we have just two monochromatic sources of the same frequency with a phase difference of ϕ, then the total pressure will be

$$p_{\text{tot}} = p_1 + p_2 e^{i\phi} \,, \tag{2.2.91}$$

so that the total RMS pressure is

$$\begin{aligned} p_{\text{tot,rms}}^2 &= p_{1,\text{rms}}^2 + p_{2,\text{rms}}^2 + \frac{2P_1 P_2}{T} \int_0^T \cos \omega t \cos(\omega t + \phi) \, \mathrm{d}t \\ &= p_{1,\text{rms}}^2 + p_{2,\text{rms}}^2 + P_1 P_2 \cos \phi \,. \end{aligned} \tag{2.2.92}$$

(Note that it's more convenient here not to use complex exponentials, since we are multiplying and need to multiply only the real parts.) The total RMS pressure due to two harmonic sources as a function of the phase difference ϕ is plotted in Fig. 2.1. As we might expect, the total pressure vanishes when the two sources are exactly out of phase (when $\phi = \pi$).

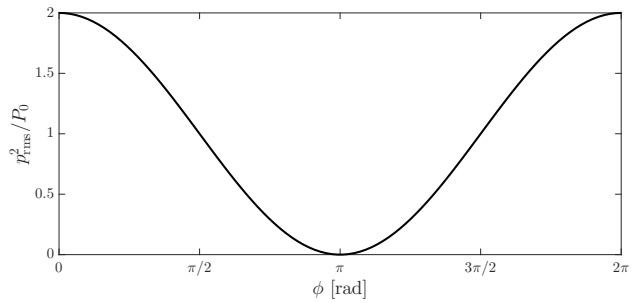


Figure 2.1: Total RMS pressure for two harmonic sources of equal strength as a function of the relative phase difference ϕ.

Finally, consider the case where we have multiple sources *of different frequencies.* What should our averaging period be then? In this case, we let our averaging interval become very long:

$$p^2_{\mathrm{rms}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T p^2 \,\mathrm{d}t \,. \tag{2.2.93}$$

We do this because it has the nice result that, as T becomes very large, it is true that

$$p^2_{\mathrm{rms}} = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\sum_n A_n \cos \omega_n t \right)^2 \mathrm{d}t = \sum \frac{A_n^2}{2} \,. \tag{2.2.94}$$

That is, the total RMS squared pressure is the sum of the RMS squared pressures of each frequency component separately. This can be thought of as a result of Parseval's theorem, which (loosely) says that the total energy (which is proportional to p^2; see the next section) must be the same in the time and frequency domains. Note also that the result is only valid when $T \gg 2\pi/\omega_{\mathrm{min}}$, where ω_{min} is the lowest frequency of interest.

Energy

The energy carried by an acoustic wave is exchanged between the potential energy stored in the fluid as the pressure wave compresses it, and the motion of the fluid particles as they oscillate. Of course energy is a finite quantity,[†] and usually we're talking about plane waves, which have infinite extent. So

[†]Exception: My niece.

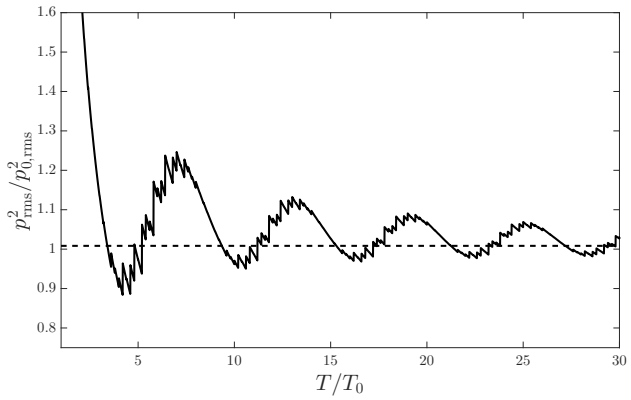


Figure 2.2: RMS pressure for two harmonic sources with $f_2/f_1 = 1.2$, computed from Eq. (2.2.94). The dashed line represents $p^2_{\mathrm{rms},1} + p^2_{\mathrm{rms},2}$ (i.e., computing the RMS pressures separately). In the large T limit, the two results approach the same value, but for smaller values of T, the difference can be significant.

usually we consider the energy density, i.e., the energy carried by the wave per unit volume. As we might guess, the kinetic energy density is simply

$$\mathcal{E}_k = \frac{1}{2}\rho_0 u^2 \,. \tag{2.2.95}$$

The potential energy may be found by considering that to compress a bit of fluid, we must do work on it. Since we've said before that acoustic propagation is adiabatic, all of this energy goes into compression of the fluid. Then we can write

$$\mathrm{d}E_p = -P\,\mathrm{d}V \,. \tag{2.2.96}$$

Since we're not losing any mass, we must have that $\varrho V = \text{const}$. Call that constant a, such that

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}\varrho} = \frac{\mathrm{d}}{\mathrm{d}\varrho}\left(\frac{a}{\varrho}\right) &= -\frac{a}{\varrho^2} \\ &= -\frac{1}{\varrho}\,\frac{a}{\varrho} = -\frac{V}{\varrho} \,. \end{aligned} \tag{2.2.97}$$

Therefore, $\mathrm{d}V = -(V/\varrho)\,\mathrm{d}\varrho$, and so

$$\mathrm{d}E_p = P\,(V/\varrho)\,\mathrm{d}\varrho \,. \tag{2.2.98}$$

But, by definition $c_0^2 = \mathrm{d}P/\mathrm{d}\varrho$ (total derivatives since we've held entropy constant), and

$$\mathrm{d}E_p = (V/\varrho\, c_0^2)\,\mathrm{d}P\,. \tag{2.2.99}$$

Integrating from some reference pressure to P and dividing by the volume gives the kinetic energy density

$$\mathcal{E}_p = \frac{1}{2}\left(\frac{p}{\rho_0 c_0}\right)^2\,. \tag{2.2.100}$$

The total energy density is then just

$$\mathcal{E} = \frac{1}{2}\rho_0\left[u^2 + \left(\frac{p}{\rho_0 c_0}\right)^2\right]\,. \tag{2.2.101}$$

If we have a progressive plane wave, so that $p = \rho_0 c_0 u$, Eq. (2.2.101) simplifies to

$$\mathcal{E} = \frac{p^2}{\rho_0 c_0^2}\,. \tag{2.2.102}$$

Intensity

The acoustic intensity measures the flow of energy per unit area. It tells us how much energy, and how quickly, the acoustic wave is carrying. The instantaneous acoustic intensity I is defined by

$$\boldsymbol{I} \equiv p\boldsymbol{u}\,. \tag{2.2.103}$$

Typically, we are interested in the intensity measured in an area normal to the propagation direction, so we will write $u = \boldsymbol{u}\cdot\boldsymbol{e}_n$, and the intensity will be a scalar. Often more useful to know is the time-averaged intensity. From Eq. (2.2.103), we have

$$I_{\mathrm{avg}} = \frac{1}{T}\int_0^T p\,u\,\mathrm{d}t. \tag{2.2.104}$$

Equation (2.2.104) has special forms for a couple of special cases.

- **Progressive Waves in Lossless Fluids**
  In this case, we have energy flow only in one direction. Therefore, $u =$

$p/\rho_0 c_0$, and we have from Eq. (2.2.104)

$$\begin{aligned} I_{\text{avg}} &= \frac{1}{\rho_0 c_0}\left(\frac{1}{T}\int_0^T p^2 \,\mathrm{d}t\right) \\ &= \frac{p_{\text{rms}}^2}{\rho_0 c_0}\,. \end{aligned} \tag{2.2.105}$$

It turns out that Eq. (2.2.105) holds even for progressive spherical waves, see Sec. A.3.

- **Harmonic Waves**
  For time harmonic waves, we can write that the pressure and particle velocity as

$$p = Pe^{-i\omega t}, \quad \text{and} \quad u = Ue^{-i\omega t}, \tag{2.2.106}$$

respectively, where P and U are the (complex) amplitudes. The intensity $I = p\,u$ is now a bit thornier since we are taking the product of quantities that we write as complex, but are interested in the real part of. We can write these quantities in polar form, i.e., $u = |U|e^{i(\theta_u - \omega t)}$, $p = |P|e^{i(\theta_p - \omega t)}$, and

$$Z = \frac{p}{u} = \frac{P}{U} = |Z|\, e^{i\phi}\,. \tag{2.2.107}$$

Computing the time-averaged intensity from Eq. (2.2.104) and using an averaging period of one cycle ($T = 2\pi/\omega$), we find

$$\begin{aligned} I_{\text{avg}} &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} (\operatorname{Re} Zu)\,(\operatorname{Re} u)\,\mathrm{d}t \\ &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \left[\operatorname{Re}\left(|Z|e^{i(\phi-\omega t)}|U|e^{i(\theta_u-\omega t)}\right)\right]\left[\operatorname{Re}\left(|U|e^{i(\theta_u-\omega t)}\right)\right]\mathrm{d}t \\ &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \Big[|Z||U|\cos\left(\theta_u+\phi-\omega t\right)\Big]\Big[|U|\cos\left(\theta_u-\omega t\right)\Big]\,\mathrm{d}t \\ &= |Z||U|^2\frac{\omega}{2\pi}\left[\frac{\pi}{\omega}\cos\phi\right] \\ &= \frac{1}{2}|U|^2|Z|\cos\phi\,. \end{aligned} \tag{2.2.108}$$

Notice that $|Z|\cos\phi = \operatorname{Re} Z$, and $\frac{1}{2}|U|^2 = u_{\text{rms}}^2$ and so

$$I_{\text{avg}} = u_{\text{rms}}^2 \operatorname{Re} Z\,. \tag{2.2.109}$$

It can also be shown that

$$I_{\text{avg}} = \frac{1}{2}\operatorname{Re}\left(PU^*\right). \tag{2.2.110}$$

Power

The sound power Π is simply the total energy per unit time delivered by the sound wave. Since the intensity gives the describes the flow of energy at a given point, the power can be computed by integrating the intensity over a surface S of interest

$$\Pi \equiv \int_S \boldsymbol{I} \cdot \boldsymbol{n} \, \mathrm{d}A, \tag{2.2.111}$$

where $\boldsymbol{n}$ is the unit normal. For plane waves in a duct with cross-sectional area A, say, then we have that $\boldsymbol{n}$ is always perpendicular to the particle velocity, and thus

$$\Pi = IA \qquad \text{(duct)}. \tag{2.2.112}$$

For outgoing spherical waves, the situation is similar, and the power is given by

$$\Pi = 4\pi r^2 I \qquad \text{(spherical waves)}. \tag{2.2.113}$$

Directivity

In some cases, the pressure field may be written

$$p(\boldsymbol{r}, t) = p(r, t)\, D(\theta, \phi), \tag{2.2.114}$$

that is, dependence of the field on the azimuth ϕ and polar θ angles may be separated from the rest of the spatial dependence. The quantity $D(\theta, \phi)$ is called the **directivity**. Note that D in Eq. (2.2.114) is *unitless*; it simply scales the pressure at a given angular position. For instance, the directivity of a baffled, circular piston with radius R in the far field turns out to be

$$D(\theta) = \frac{J_1 \left(kR \sin\theta\right)}{kR \sin\theta} \tag{2.2.115}$$

(see Sec. 2.5.4 for derivation). Note there is no ϕ-dependance, since the problem is axially symmetric.

2.3 Measurement and Signal Processing

2.3.1 The Frequency Domain

Fourier's theorem says that any function[†] can be represented as a sum of sinusoidal functions. The Fourier series or transform of a function $\tilde{f}(\omega)$ tells us the amplitude and phase of the component of $f(t)$ with angular frequency ω.

Fourier Series

If the signal is of finite duration T, then it can be represented by its Fourier series. The series is defined

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{T} + b_n \sin\frac{n\pi t}{T}\,, \tag{2.3.1}$$

where the coefficients are given by

$$a_n = \frac{1}{T/2}\int_0^T s(t)\,\cos\frac{n\pi t}{T}\,\mathrm{d}t \quad \text{and} \tag{2.3.2}$$

$$b_n = \frac{1}{T/2}\int_0^T s(t)\,\sin\frac{n\pi t}{T}\,\mathrm{d}t\,. \tag{2.3.3}$$

For most cases of interest, recorded signals will be sampled at finite, uniform time intervals Δt. Thus the summation in Eq. (2.3.1) need not be carried out over all frequencies. The signal may be reconstructed by including only frequencies up to $1/2\Delta t$.[‡]

Fourier Transforms

If a signal is infinite in extent, then the summations in the Fourier series become integrals, and we obtain the Fourier transform. Note that we could write Eq. (2.3.1) as the product of a complex coefficient and complex exponential. Using this form the limiting case for the complex coefficients becomes

$$\tilde{s}(\omega) = \mathcal{F}[s(t)] \equiv \int_{-\infty}^{\infty} s(t)\,e^{i\omega t}\,\mathrm{d}t \tag{2.3.4}$$

[†] One could of course construct a function whose Fourier transform is not well-defined. But since its relevance to acoustics is in its applications to functions representing acoustic pressure and velocity, we assume these functions will always be continuous, bounded, and square integrable.

[‡] This is the Nyquist-Shannon theorem. The details and proof are probably out of scope for the acoustics exam, but it is worth knowing about the relationship $f_{\max} = 1/2\Delta t$. If you're interested, a pretty accessible proof is given by Todd K. Moon in his lecture notes.

The function $\tilde{s}(\omega)$ assigns to every frequency ω (not just discrete multiples of $1/\Delta t$) a complex number. The magnitude $\|\tilde{s}(\omega)\|$ gives the amplitude of the sinusoidal signal with that frequency, and the argument $\angle\tilde{s}(\omega)$ gives the phase of the signal. Fourier's theorem tells us that if we add up the sinusoids of all frequencies ω with the amplitudes and phases assigned to them by the function $\tilde{s}(t)$, we will recover our original signal $s(t)$.

Note what happens if we assume that the pressure is harmonic, and then take the Fourier transform of the linear wave equation Eq. (2.2.41):

$$\begin{aligned} \mathcal{F}\left[\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2}\right] &= 0 \\ \nabla^2\tilde{p} - \frac{1}{c_0^2}\mathcal{F}\left[\frac{\partial^2 p}{\partial t^2}\right] &= 0\,. \end{aligned} \tag{2.3.5}$$

Note for the first term that the spatial derivative does not depend on time, so we may pull it out of the integration. Then using the derivative property of the transform (see Sec. 3.5.3), we have have

$$\begin{aligned} \nabla^2\tilde{p} - \frac{i^2\omega^2}{c_0^2}\tilde{p} &= 0 \\ \left(\nabla^2 + k^2\right)\tilde{p} &= 0\,. \end{aligned} \tag{2.3.6}$$

Equation (2.3.6) is called the Helmholtz equation, and is equivalent to the wave equation for waves containing only a single frequency.

Mind the Signs Typically the sign of the exponent in Eq. (2.3.4) doesn't matter as long as we use the opposite sign for the inverse transform. However, since we have used the time convention that harmonic signals are proportional to $e^{-i\omega t}$, we must use the ***positive*** sign for the forward transform. Our sign convention also means that we've chosen a positive sign for the spatial dependence of progressive plane waves, that is, $p \propto e^{+ikx}$. So, if we were to do a spatial transform of the signal, this forward spatial transform must use the ***negative*** sign in the exponent

$$\tilde{s}(k) = \mathcal{F}_k[s(x)] \equiv \int_{-\infty}^{\infty} s(x)\, e^{-ikx}\,\mathrm{d}x\,. \tag{2.3.7}$$

2.4 Reflection and Transmission

This section will deal with the behavior of plane waves incident on the boundary between two fluids, where in a few cases the half-spaces are separated by a

boundary. The fundamental approach for handling impedance discontinuities is satisfaction of the boundary conditions. For free surfaces, the pressure and normal particle velocity be continuous at the interface. For the cases of an intervening boundary, we'll set up a force balance to account for any pressure differetials.

2.4.1 Planar Interface

It is convenient to define the pressure reflection coefficient as the ratio between the reflected and incident (acoustic) pressure

$$\mathcal{R} \equiv \frac{p_{\text{ref}}}{p_{\text{inc}}} \,. \tag{2.4.1}$$

And similarly the, transmitted coefficient as the ration of the transmitted pressure to that incident:

$$\mathcal{T} \equiv \frac{p_{\text{trans}}}{p_{\text{inc}}} \,. \tag{2.4.2}$$

Now we must have that the pressure must be equal on both sides of the interface between the two fluids.[†] Then,

$$\begin{aligned} p_{\text{inc}} + p_{\text{ref}} &= p_{\text{trans}} \Big|_{x=0} \\ \implies 1 + \mathcal{R} &= \mathcal{T} \,. \end{aligned} \tag{2.4.3}$$

We also must have that the normal velocity matches at the interface—otherwise the fluids would need to separate or pass through each other. Since we have plane waves $u = \pm p/Z$, and so

$$\begin{aligned} u_{\text{inc}} + u_{\text{ref}} &= u_{\text{trans}} \Big|_{x=0} \\ \frac{1}{Z_1} p_{\text{inc}} - \frac{1}{Z_1} p_{\text{ref}} &= \frac{1}{Z_2} p_{\text{trans}} \\ \implies 1 - \mathcal{R} &= \frac{Z_1}{Z_2} \mathcal{T} \,. \end{aligned} \tag{2.4.4}$$

Adding Eqs. (2.4.3) and (2.4.4) gives

$$\begin{aligned} 2 &= (1 + Z_1/Z_2)\mathcal{T} \\ \implies & \boxed{\mathcal{T} = \frac{2Z_2}{Z_1 + Z_2} \,.} \end{aligned} \tag{2.4.5}$$

[†] "[O]therwise the interface, being massless, would experience an infinite acceleration."—Ref. 4.

Then substituting Eq. (2.4.5) into Eq. (2.4.3) allows us to solve for the reflection coefficient

$$\begin{aligned} 1 + \mathcal{R} &= \frac{2Z_2}{Z_1 + Z_2} \\ \implies &\boxed{\mathcal{R} = \frac{Z_2 - Z_1}{Z_1 + Z_2}\,.} \end{aligned} \tag{2.4.6}$$

Power and Intensity Equations (2.4.5) and (2.4.6) indicate the fraction of the pressure that is transmitted or reflected from an interface. But how about intensity, or equivalently, what fraction of the incident power[†] makes it through the interface? Well, for progressive waves the (time averaged) intensity is given by Eq. (2.2.105), modified here as

$$I = \frac{P^2}{2\rho_0 c_0}\,, \tag{2.4.7}$$

where $P = \sqrt{2}p_{\text{rms}}$ is the pressure amplitude. The intensity reflection coefficient is then defined as we might expect

$$\begin{aligned} r \equiv \frac{I_{\text{ref}}}{I_{\text{inc}}} &= \frac{P_{\text{ref}}^2/2\rho_1 c_1}{P_{\text{inc}}^2/2\rho_1 c_1} \\ &= \frac{(\mathcal{R}P_{\text{inc}})^2}{P_{\text{inc}}^2} \\ &= \mathcal{R}^2\,. \end{aligned} \tag{2.4.8}$$

Similarly for the transmitted intensity,

$$\begin{aligned} \tau \equiv \frac{I_{\text{trans}}}{I_{\text{inc}}} &= \frac{P_{\text{trans}}^2/2\rho_1 c_1}{P_{\text{inc}}^2/2\rho_2 c_2} \\ &= \frac{Z_2}{Z_1}\,\frac{(\mathcal{T}P_{\text{inc}})^2}{P_{\text{inc}}^2} \\ &= \frac{Z_2}{Z_1}\,\mathcal{T}^2\,. \end{aligned} \tag{2.4.9}$$

Since we've included no loss mechanisms, we must have conservation of energy, i.e.,

$$r + \tau = 1\,. \tag{2.4.10}$$

[†]Strictly, the power fraction is not the same as the intensity fraction, since $W = \int I \cdot \boldsymbol{e}_n \,\mathrm{d}A$. Thus if there is a change in area (e.g., if a duct widens suddenly), this transmitted power fraction will be the transmitted intensity fraction multiplied by A_2/A_1.

Oblique Incidence

The derivation of reflection and transmission coefficients in the previous section assumed that the waves were normally incident on the interface. But what if they are incident at an angle instead? In this case call the x-axis the one normal to the interface, and then write the incident wave as

$$p_{\text{inc}} = A_1 e^{i(k_{1x}x + k_{1y}y - \omega t)}, \tag{2.4.11}$$

where $k_{1x} = k_1 \cos\theta_{\text{inc}}$, $k_{1y} = k_1 \sin\theta_{\text{inc}}$, and θ_{inc} is the angle the wave makes with the x-axis (i.e., the normal); see Fig. 2.3. Some conventions prefer to use the grazing angle, that is, the angle made with the interface. The conversion is not too difficult, but only the normal-incidence convention will be used here.

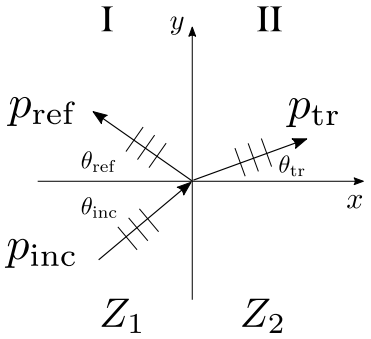


Figure 2.3: Geometry for oblique incidence on a planar interface.

We still must have continuity of pressure and particle velocity in the x-direction. Then at the interface ($x = 0$),

$$\begin{aligned} p_{\text{inc}} + p_{\text{ref}} &= p_{\text{tr}} \\ A_1 e^{ik_{1y}\sin\theta_{\text{inc}}} + B_1 e^{-ik_{1y}\sin\theta_{\text{ref}}} &= A_2 e^{ik_{2y}\sin\theta_{\text{tr}}}. \end{aligned} \tag{2.4.12}$$

Now, since Eq. (2.4.12) must be true for all y, we conclude that the exponents must all be equal. Then, we see that our boundary conditions yield a statement of Snell's law:

$$\theta_{\text{inc}} = \theta_{\text{ref}}, \tag{2.4.13}$$

and

$$\frac{\sin\theta_{\text{inc}}}{c_1} = \frac{\sin\theta_{\text{ref}}}{c_1} = \frac{\sin\theta_{\text{tr}}}{c_2} \tag{2.4.14}$$

Then, we can compute the transmission angle from

$$\theta_{\text{tr}} = \arcsin\left(\frac{c_2}{c_1}\,\sin\theta_{\text{inc}}\right). \tag{2.4.15}$$

Now the pressure has to match at the boundary. Since we decided the exponents of all terms in Eq. (2.4.12) are equal, then we have as before

$$1 + \mathcal{R} = \mathcal{T}\,, \tag{2.4.16}$$

just as in the normally-incident case. Now the normal component of incident particle velocity will be

$$\begin{aligned} u_{x,\text{inc}} = \boldsymbol{u}\cdot\boldsymbol{e}_x &= \frac{k_x}{i\omega\rho_0}Ae^{i\left(k_x x + k_y y\right)} \\ &= \frac{p_{\text{inc}}}{Z_1}\cos\theta_{\text{inc}}\,. \end{aligned} \tag{2.4.17}$$

The results are similar for the transmitted and reflected wave. Then, since we need to match the particle velocity at the interface, we write

$$\begin{aligned} u_{x,\text{inc}} + u_{x,\text{ref}} &= u_{x,\text{tr}} \\ \left(\frac{p_{\text{inc}}}{Z_1}\cos\theta_{\text{inc}}\right) - \left(\frac{p_{\text{ref}}}{Z_1}\cos\theta_{\text{ref}}\right) &= \left(\frac{p_{\text{tr}}}{Z_2}\cos\theta_{\text{tr}}\right) \\ 1 - \mathcal{R} &= \frac{Z_1/\cos\theta_{\text{inc}}}{Z_2/\cos\theta_{\text{tr}}}\mathcal{T}\,. \end{aligned} \tag{2.4.18}$$

Note the quantity

$$Z_{\text{ac}} = Z/\cos\theta\,. \tag{2.4.19}$$

This sometimes called the "acoustic impedance" and is very useful since, when Eqs. (2.4.16) and (2.4.18) are solved, we would get

$$\boxed{\mathcal{R} = \frac{Z_{\text{ac},2} - Z_{\text{ac},1}}{Z_{\text{ac},2} + Z_{\text{ac},1}}} \tag{2.4.20}$$

which matches very neatly the plane wave result. The transmission coefficient is then obtained from Eq. (2.4.16).

But hold on, it's conceivable that the argument of Eq. (2.4.15) would have a magnitude greater than 1. What happens then? In this case, we'll use the

Pythagorean identity to write[†]

$$\begin{aligned}
\cos^2\theta_{\text{tr}} &= 1-\sin^2\theta_{\text{tr}} \\
\implies \cos\theta_{\text{tr}} &= +\sqrt{1-\sin^2\theta_{\text{tr}}} \\
&= \sqrt{1-\left(\frac{c_2}{c_1}\right)^2\sin^2\theta_{\text{inc}}} \\
&= \underbrace{i\sqrt{\left(\frac{c_2}{c_1}\right)^2\sin^2\theta_{\text{inc}}-1}}_{\equiv\zeta} \,. \qquad (2.4.21)
\end{aligned}$$

With Eq. (2.4.21) we can write the transmitted wave as

$$\begin{aligned}
p_{\text{tr}} &= A_2\exp\left[i\big(\underbrace{k_2 y\sin\theta_{\text{tr}}}_{k_1 y\sin\theta_{\text{inc}}}+k_2 x\underbrace{\cos\theta_{\text{tr}}}_{\zeta}-\omega t\big)\right] \\
&= A_2 e^{ik_2\zeta x}e^{i(k_1 y\sin\theta_{\text{inc}}-\omega t)} \,. \qquad (2.4.22)
\end{aligned}$$

Now there's a few cases of interest.

- $c_1 > c_2$
  If the second medium is slower than the first, we see from Eq. (2.4.21) that ζ is guaranteed to be real, and then that we will have a wave propagating at a lower angle in the second medium. That is, the wave will refract toward the normal. Sound seeks low speed!

- $c_2 > c_1$
  If the second a bit faster than the first, we have two further possibilities. First, define the critical angle

$$\theta_c = \arcsin\frac{c_1}{c_2} \,. \qquad (2.4.23)$$

  – $\theta_{\text{inc}} < \theta_c$
    If the wave is incident below the critical angle, ζ remains real, and we will still have regular refraction. This time the wave will bend away from the normal.
  – $\theta_{\text{inc}} > \theta_c$
    If the incident angle is above the critical angle, then we see that ζ becomes purely imaginary. Thus the term $e^{i\zeta x}$ in Eq. (2.4.22)

[†]We'll choose the positive root for reasons that will hopefully be clearer in a moment.

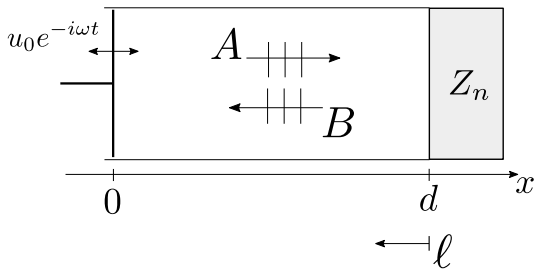


Figure 2.4: Geometry for a tube driven by a harmonic piston at $x = 0$ and terminated with a load impedance Z_n at $x = d$.

becomes $e^{-\alpha x}$, where α is a real number. The transmitted wave then propagates *only* in the y-direction, and decays exponentially in the x-direction.[†] These waves which "cling"[6] to the interface are called "evanescent waves".

2.4.2 Standing Waves

Consider a pipe of length d filled with a fluid with $Z_0 = \rho_0 c_0$, shown in Fig. 2.4. The pipe also has some impedance termination Z_n and is driven with a harmonic piston at $x = 0$. We'll assume low frequency so that we have only 1D propagation.[‡] The resulting field will have the form (taking the $e^{-i\omega t}$ dependence as implied)

$$p(x) = Ae^{ikx} + Be^{-ikx} \tag{2.4.24}$$

$$u(x) = \frac{A}{Z_0}\, e^{ikx} - \frac{B}{Z_0}\, e^{-ikx}\,. \tag{2.4.25}$$

It turns out its convenient to write the pressure and velocity as functions of distance from the end. So write $\ell = d - x$, so that Eq. (2.4.24) and Eq. (2.4.25) becomes

$$p(\ell) = A'e^{-ik\ell} + B'e^{ik\ell} \tag{2.4.26}$$

$$u(\ell) = \frac{A'}{Z_0}\, e^{-ik\ell} - \frac{B'}{Z_0}\, e^{ik\ell}\,, \tag{2.4.27}$$

[†]Note that if we had taken the negative root in Eq. (2.4.21), we would have had exponential growth in the amplitude of the transmitted wave as we moved away from the interface. This would be unphysical, and so the positive root was chosen.

[‡]That is, the wavelength must be large compared to the radius of the tube. For a rigid tube, we must have $ka \ll \alpha'_{01} \approx 1.841$.

where $A' = Ae^{ikd}$ and $B' = Be^{-ikd}$ are new arbitrary constants. Then using the normal incidence reflection coefficient $\mathcal{R}$ [Eq. (2.4.6)], we can write

$$p(\ell) = A' \left(e^{-ik\ell} + \mathcal{R}\, e^{ik\ell}\right) \tag{2.4.28}$$

$$u(\ell) = \frac{A'}{Z_0} \left(e^{-ik\ell} - \mathcal{R}\, e^{ik\ell}\right), \tag{2.4.29}$$

where

$$\mathcal{R} = \frac{Z_n - Z_0}{Z_n + Z_0}. \tag{2.4.30}$$

Finally we need to match the boundary condition at $\ell = d$. Since at $\ell = d$, $u = u_0$, we have from Eq. (2.4.27)

$$\begin{aligned} u_0 &= \frac{A'}{Z_0} \left(e^{-ikd} - \mathcal{R}\, e^{ikd}\right) \\ &\implies A' = \rho_0 c_0 u_0 \left(e^{-ikd} - \mathcal{R}\, e^{ikd}\right)^{-1} \end{aligned} \tag{2.4.31}$$

The velocity field in the tube is then, from Eqs. (2.4.29) and (2.4.31)

$$u(x) = u_0 \frac{e^{-ik(d-x)} - \mathcal{R}\, e^{ik(d-x)}}{e^{-ikd} - \mathcal{R}\, e^{ikd}}, \tag{2.4.32}$$

Some standing wave patterns for a rigid, free, and intermediate terminating impedance is shown in Fig. 2.5.

Note the impedance seen at any point in the tube is

$$Z(\ell) = \frac{p(\ell)}{u(\ell)} = Z_0 \frac{e^{-ik\ell} + \mathcal{R}\, e^{ik\ell}}{e^{-ik\ell} - \mathcal{R}\, e^{ik\ell}}. \tag{2.4.33}$$

We might notice that Eq. (2.4.33) could well vanish. In this limit, the piston would feel no force from the fluid in the tube. This condition is called *resonance.*

2.4.3 Multiple Layers

Suppose we have a layer of thickness d separating two half-spaces. Call the layer region II and the half-spaces I and III (see Fig. 2.6). Since there will be no waves travelling leftward (from infinity) in region III, we can write the pressure field in each as

$$\text{I}: \quad p_1 = A_1 e^{i(k_1 x - \omega t)} + B_1 e^{-i(k_1 x + \omega t)} \tag{2.4.34}$$

$$\text{II}: \quad p_2 = A_2 e^{i(k_2 x - \omega t)} + B_2 e^{-i(k_2 x + \omega t)} \tag{2.4.35}$$

$$\text{III}: \quad p_3 = A_3 e^{i(k_3 x - \omega t)}. \tag{2.4.36}$$

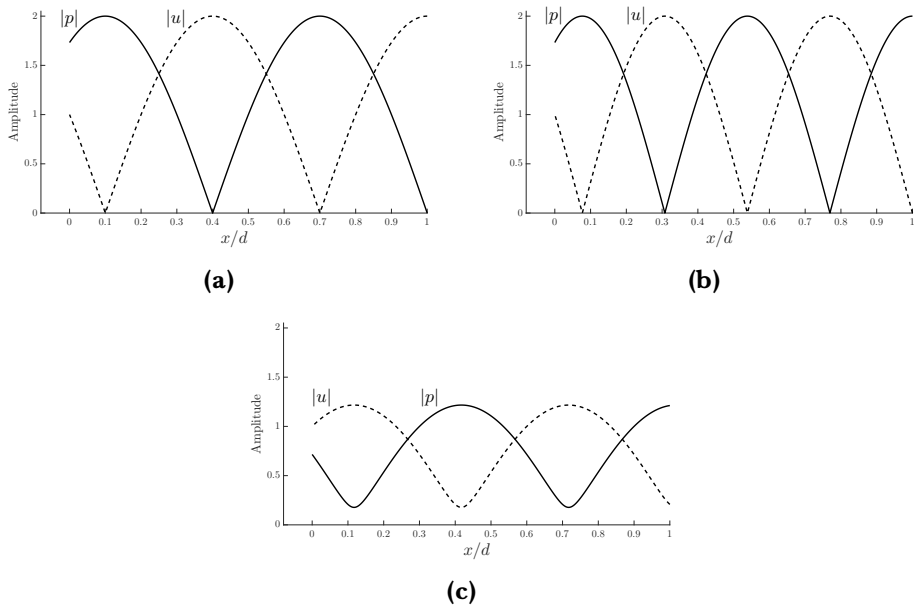


Figure 2.5: Plots of the pressure (solid line) and particle velocity (dashed lines) amplitudes in terminated tubes. **(a)** Free case, where $Z_n = 0$. Note that the pressure vanishes at the termination. **(b)** Rigid termination, so that $Z_n \to \infty$. In this case the particle velocity is 0 at $x = d$. **(c)** Here, $Z = Z_0(5 + 3i)$.

Since the time dependence $e^{-i\omega t}$ is the same for all terms, we'll take it as implied for the rest of this section to save some ink.

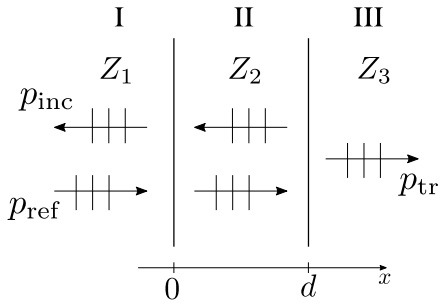


Figure 2.6: Geometry for the three-medium reflection and transmission problem.

As for any acoustics problem, we need to satisfy the boundary conditions. As

before in the planar interface, the pressure to match at $x = 0$ (i.e. the interface between regions I and II):

$$\begin{aligned} p_1 &= p_2\Big|_{x=0} \\ \implies A_1 + B_1 &= A_2 + B_2\,. \end{aligned} \tag{2.4.37}$$

Since we normally-incident have plane waves, $u = \pm p/Z$, and so the velocity matching condition is

$$\begin{aligned} u_1 &= u_2\Big|_{x=0} \\ \implies (A_1 - B_1)Z_1^{-1} &= (A_2 - B_2)Z_2^{-1} \\ \implies (A_1 - B_1) &= \frac{Z_1}{Z_2}(A_2 - B_2)\,. \end{aligned} \tag{2.4.38}$$

The boundary conditions must then be applied to the interface at $x = d$. It was super convenient that $x = 0$ at the first interface, since it caused the exponential term to go to 1. We can get some of this benefit if we write the pressure in region 3 in terms of a shifted coordinate $\xi = x - d$,

$$p_3 = A_3' e^{ik_3\xi}\,. \tag{2.4.39}$$

Then, matching the pressure and velocity at $x = d$ gives

$$\begin{aligned} p_2 &= p_3\Big|_{x=d} \\ \implies A_2 e^{ik_2 d} + B_2 e^{-ik_2 d} &= A_3'\,, \end{aligned} \tag{2.4.40}$$

and

$$\begin{aligned} u_2 &= u_3\Big|_{x=d} \\ \implies A_2 e^{ik_2 d} - B_2 e^{-ik_2 d} &= \frac{Z_2}{Z_3}A_3'\,, \end{aligned} \tag{2.4.41}$$

respectively. Now, adding Eq. (2.4.40) and Eq. (2.4.41) to eliminate B_2 gives

$$A_2 = \frac{1}{2}\left(1 + \frac{Z_2}{Z_3}\right)A_3' e^{-ik_2 d}\,, \tag{2.4.42}$$

and subtracting to eliminate A_2 gives

$$B_2 = \frac{1}{2}\left(1 - \frac{Z_2}{Z_3}\right)A_3' e^{ik_2 d}\,, \tag{2.4.43}$$

Now, add Eq. (2.4.37) to Eq. (2.4.38) to eliminate B_1:

$$A_1 = \frac{1}{2}\left(1 + Z_1/Z_2\right) A_2 + \frac{1}{2}\left(1 - Z_1/Z_2\right) B_2 \tag{2.4.44}$$

Finally, substitute Eqs. (2.4.42) and (2.4.43) into Eq. (2.4.44) and solve for A_3'/A_1

$$\begin{aligned} A_1 &= \frac{1}{2}\left(1 + Z_1/Z_2\right)\left[\frac{1}{2}\left(1 + Z_2/Z_3\right) A_3' e^{-ik_2 d}\right] + \\ &\qquad \frac{1}{2}\left(1 - Z_1/Z_2\right)\left[\frac{1}{2}\left(1 - Z_2/Z_3\right) A_3' e^{ik_2 d}\right] \\ \implies 4\,\frac{A_1}{A_3'} &= \left(1 + Z_1/Z_2 + Z_2/Z_3 + Z_1/Z_3\right) e^{-ik_2 d} + \\ &\qquad \left(1 - Z_1/Z_2 - Z_2/Z_3 + Z_1/Z_3\right) e^{ik_2 d} \\ &= 2\left(1 + Z_1/Z_3\right)\cos kd - 2i\left(Z_1/Z_2 + Z_2/Z_3\right)\sin kd \\ \implies \boxed{\mathcal{T} = \frac{2}{\left(1 + Z_1/Z_3\right)\cos k_2 d - i\left(Z_1/Z_2 + Z_2/Z_3\right)\sin k_2 d}\,.} \end{aligned} \tag{2.4.45}$$

To find the reflection coefficient $\mathcal{R} \equiv B_1/A_1$, substitute Eqs. (2.4.42) and (2.4.43) into Eq. (2.4.37) to find

$$\begin{aligned} A_1 + B_1 &= \left[\frac{1}{2}\left(1 + \frac{Z_2}{Z_3}\right) A_3' e^{-ik_2 d}\right] + \left[\frac{1}{2}\left(1 - \frac{Z_2}{Z_3}\right) A_3' e^{ik_2 d}\right] \\ \implies 2(1 + \mathcal{R}) &= \left[(1 + Z_2/Z_3)e^{-ikd} + (1 - Z_2/Z_3)e^{ik_2 d}\right]\mathcal{T} \\ &= \left[2\cos k_2 d - 2i(1 + Z_2/Z_3)\sin kd\right]\mathcal{T}\,. \end{aligned} \tag{2.4.46}$$

Then using the transmission coefficient we found [Eq. (2.4.45)], we have from Eq. (2.4.46)

$$\begin{aligned} \mathcal{R} &= \frac{2\cos k_2 d - 2i(1 + Z_2/Z_3)\sin k_2 d}{\left(1 + Z_1/Z_3\right)\cos k_2 d - i\left(Z_1/Z_2 + Z_2/Z_3\right)\sin k_2 d} - 1 \\ \implies \boxed{\mathcal{R} &= \frac{\left(1 - Z_1/Z_3\right)\cos k_2 d - i(Z_2/Z_3 - Z_1/Z_2)\sin k_2 d}{\left(1 + Z_1/Z_3\right)\cos k_2 d - i\left(Z_1/Z_2 + Z_2/Z_3\right)\sin k_2 d}\,.} \end{aligned} \tag{2.4.47}$$

Note that if we let the layer become vanishingly thin ($k_2 d \to 0$), Eqs. (2.4.45) and (2.4.47) reduce to the expressions we obtained for the two-medium problem (see Sec. 2.4.1).

2.4.4 Radiation and Transmission for Non-Fluids

Radiation from Flexural Waves

Consider a right-travelling wave with frequency ω on a plate, which travels in the positive x direction with velocity c_{pl},[†] as shown in Fig. 2.7 Now we require that the normal particle velocity of the plate match that in the fluid (air).[‡] The vertical displacement ξ of the surface can be written

$$\xi = U_0 e^{-i\omega\left(t - x/c_{\text{pl}}\right)}, \tag{2.4.48}$$

and so its (normal) velocity is

$$\dot{\xi} = -i\omega U_0 e^{-i\omega\left(t - x/c_{\text{pl}}\right)}\,. \tag{2.4.49}$$

Now considering the field in air above the plate, for a plane wave traveling with some wavevector $\boldsymbol{k}$,

$$\begin{aligned} p^+ &= A e^{i(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)} \\ &= A e^{i(k x_x + k_z z - \omega t)}, \end{aligned} \tag{2.4.50}$$

where $k_x \equiv k\sin\theta$, $k_z \equiv k\cos\theta$, and θ is the angle that $\boldsymbol{k}$ makes with the z axis. Similarly, the field below the plate is

$$p^- = A e^{i(k x_x - k_z z - \omega t)}, \tag{2.4.51}$$

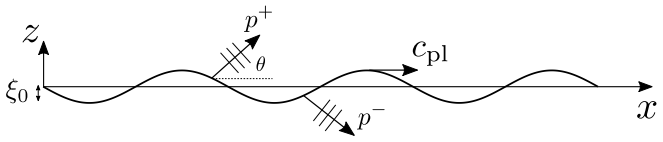


Figure 2.7: Geometry for the acoustic field generated by the flexural wave in a thin plate.

[†]And we'll assume that the plane is effectively infinite, no energy will propagate back to the left (for $x > x_0$).

[‡]The velocity of the surface will also have a component in the horizontal (x) direction. But we're assuming that the air is inviscid, i.e., that it cannot support shear motion. Therefore, this horizontal motion of the surface won't contribute to the acoustic radiation.

We can obtain the normal particle velocity from the Euler equation [Eq. (2.2.34)]

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} &= -\frac{1}{\rho_0}\nabla p \\ \implies \frac{\partial u_z}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \\ i\omega u_z &= -\frac{1}{\rho_{\text{air}}}\frac{\partial p}{\partial z} \end{aligned} \tag{2.4.52}$$

Then for the field above we have

$$\begin{aligned} u_z^+ &= -\frac{1}{i\omega\rho_0}\left(jk_z p^+\right) \\ &= -\frac{k\cos\theta}{\omega\rho_0}p \\ &= \frac{\cos\theta}{\rho_0 c_0}p, \end{aligned} \tag{2.4.53}$$

and similarly below

$$u_z^- = -\frac{\cos\theta}{\rho_0 c_0}p\,. \tag{2.4.54}$$

Matching the particle velocity above the plate to the normal velocity of the plate gives [from Eqs. (2.4.49) and (2.4.53)]

$$\begin{aligned} u_z^+ &= \dot{\xi}\Big|_{z=0} \\ \frac{\cos\theta}{\rho_0 c_0}p &= -i\omega U_0 e^{-i\omega\left(t-x/c_{\text{pl}}\right)} \\ \frac{\cos\theta}{\rho_0 c_0}Ae^{ikx\sin\theta} &= -i\omega U_0 e^{i\omega x/c_{\text{pl}}}. \end{aligned} \tag{2.4.55}$$

Now the left- and right-hand sides of Eq. (2.4.55) are complex. In order for the equality to hold always, their magnitudes and phases must match uniformly. Then, we can say that

$$\begin{aligned} \frac{\cos\theta}{\rho_0 c_0}A &= -i\omega U_0 \\ \implies A &= -\frac{i\omega\rho_0 c_0 U_0}{\cos\theta}, \end{aligned} \tag{2.4.56}$$

and

$$\begin{aligned} jkx\sin\theta &= i\omega\frac{x}{c_{\mathrm{pl}}} \\ \implies \frac{\omega}{c_0}\sin\theta &= \frac{\omega}{c_{\mathrm{pl}}} \\ \implies \sin\theta &= \frac{c_0}{c_{\mathrm{pl}}}\,. \end{aligned} \tag{2.4.57}$$

Now since

$$\begin{aligned} \cos\theta &= \sqrt{1-\sin^2\theta} \\ &= \sqrt{1-c_0^2/c_{\mathrm{pl}}^2} \equiv \gamma \end{aligned} \tag{2.4.58}$$

substitution of Eq. (2.4.56) into Eq. (2.4.50) and use of Eqs. (2.4.57) and (2.4.58) gives

$$\begin{aligned} p^+ &= -\frac{i\omega\rho_0 c_0 U_0}{\cos\theta}e^{i(kx\sin\theta+kz\cos\theta-\omega t)} \\ &= -\frac{i\omega\rho_0 c_0 U_0}{\gamma}e^{i\left(kxc_0/c_{\mathrm{pl}}+kz\gamma-\omega t\right)} \\ &= -\frac{i\omega\rho_0 c_0 U_0}{\gamma}e^{i\omega\left[\frac{x}{c_{\mathrm{pl}}}+\frac{z}{c_0}\gamma-t\right]}, \end{aligned} \tag{2.4.59}$$

and

$$p^- = -\frac{i\omega\rho_0 c_0 U_0}{\gamma}e^{i\omega\left[\frac{x}{c_{\mathrm{pl}}}-\frac{z}{c_0}\gamma-t\right]}\,. \tag{2.4.60}$$

To find the horizontal components of the particle velocity, define the shorthands

$$A' \equiv \frac{i\omega\rho_0 c_0 U_0}{\gamma} \tag{2.4.61}$$

$$\phi_\pm \equiv i\omega\left[\frac{x}{c_{\mathrm{pl}}} \pm \frac{z}{c_0}\gamma - t\right], \tag{2.4.62}$$

such that, from Eq. (2.4.52) and Eq. (2.4.59)

$$\begin{aligned} u_x^+ &= -\frac{1}{i\omega\rho_0}\frac{\partial p^+}{\partial x} \\ &= -\frac{1}{i\omega\rho_0}\frac{\partial}{\partial x}\left(A' e^{\phi_+}\right) \\ &= -\frac{1}{i\omega\rho_0}\underbrace{A' e^{\phi_+}}_{-p^+}\frac{\partial \phi_+}{\partial x} \\ &= \frac{1}{i\omega\rho_0} p^+ \left(\frac{i\omega}{c_{\text{pl}}}\right) \\ &= \frac{1}{\rho_0 c_{\text{pl}}} p^+, \end{aligned} \tag{2.4.63}$$

and similarly below the plate,

$$\begin{aligned} u_x^- &= -\frac{1}{i\omega\rho_0}\frac{\partial p^-}{\partial x} \\ &= \frac{1}{\rho_0 c_{\text{pl}}} p^- . \end{aligned} \tag{2.4.64}$$

So we've solved for our pressure and velocity fields. But what do they look like? First suppose $c_{\text{pl}} < c_0$, so that γ becomes purely imaginary. Thus, we expect propagation to be evanescent in z, and that's exactly what we see in the first case Fig. 2.8. When $c_{\text{pl}} = c_0$, then $\gamma = 0$, and propagation is in the x direction only (center of Fig. 2.8).

Now if the plate speed is a bit higher, say $c_{\text{pl}}/c_0 = 1.1$, we see that the z wavenumber becomes real, and we will have radiation into the medium. This is borne out in the rightmost of Fig. 2.8. As c_{pl} becomes large, the radiated angle will become steeper. In the limit that c_{pl} becomes very large, the wave will be radiated nearly vertically, perpendicular to the plate.

Transmission through Panels

We might imagine a plate (like the one whose radiation was discussed in the previous section) has two extremes in terms of its acoustical behavior. On one end of the range, it may be very light, such that the amount of sound that gets through is determined by how stiff the panel is (e.g., a thin sheet of metal). In this case, transmission is *stiffness dominated.* At the other extreme, the "plate" might be totally flexible, but quite dense (e.g., a heavy curtain). In this case,

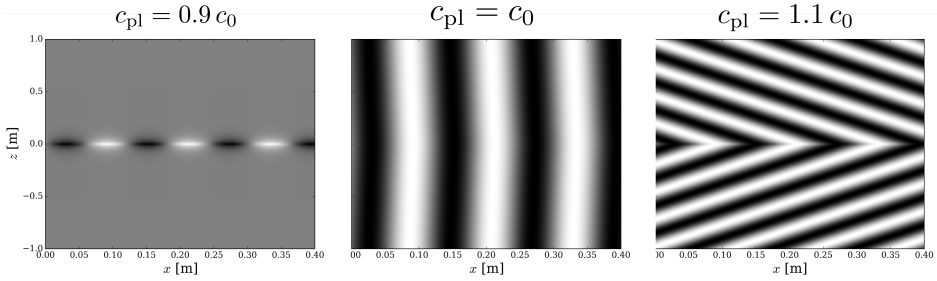


Figure 2.8: Pressure fields radiated from a flexural wave in a plate. In the first case (left), the plate wave speed is less than the surrounding medium, and the field is evanescent. In the middle case, the wave speeds are equal and the wave travels parallel to the plate surface. When $c_{\mathrm{pl}} > c_0$ radiation is at an angle to the plate, which increases with c_{pl}.

transmission is *mass dominated*. We'll discuss this case first.[†]

The Mass Law To obtain expressions for reflection and transmission through the curtain,[‡] start by writing Newton's second law at the curtain interface (since the interface is no longer massless, pressure continuity is no longer required)

$$m_{\mathrm{curtain}}\dot{v}_{\mathrm{curtain}} = p_{\mathrm{inc}} + p_{\mathrm{ref}} - p_{\mathrm{tr}}\,, \tag{2.4.65}$$

where v_{curtain} is the normal velocity of the curtain, and m_{curtain} is the curtain's mass per unit area (or acoustic mass). Now From Fig. 2.9 we see that the incident sound wave will drive flexural waves along the length of the interface. Since the medium is the same on both sides of the curtain, we have that $\theta_{\mathrm{inc}} = \theta_{\mathrm{tr}}$. We also note that wavelengths of the sound wave and the flexural wave are related by

$$\sin\theta_{\mathrm{inc}} = \frac{\lambda_{\mathrm{inc}}}{\lambda_{\mathrm{trace}}}\,. \tag{2.4.66}$$

As was derived in the previous section, the velocity of the curtain must match

[†] All analysis in this section will assume a time-harmonic signal, i.e. $p \propto e^{-i\omega t}$.

[‡] We'll assume that there is air on either side of the curtain. The derivation could proceed with different media, but it's hard to imagine such a situation would have much practical interest.

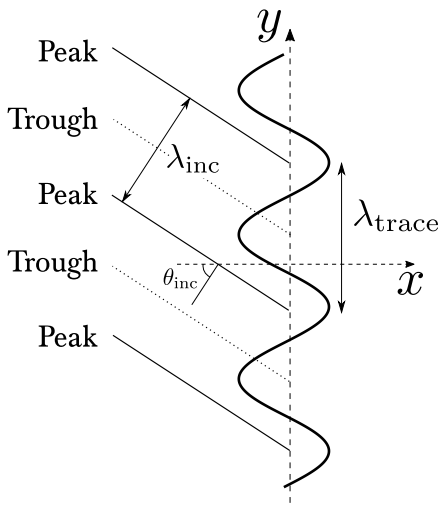


Figure 2.9: Oblique incidence on a massive curtain.

the particle velocity in the x-direction of the transmitted wave [see Eq. (2.4.53)]

$$\begin{aligned} v_{\text{curtain}} = u_{x,\text{tr}} &= \frac{\cos\theta}{\rho_0 c_0} p_{\text{tr}} \\ \implies \dot{v}_{\text{curtain}} &= \frac{-i\omega\cos\theta}{\rho_0 c_0} p_{\text{tr}}\,, \end{aligned} \tag{2.4.67}$$

and so Eq. (2.4.65) becomes

$$\begin{aligned} m_{\text{curtain}} \left(\frac{-i\omega\cos\theta}{\rho_0 c_0} p_{\text{tr}} \right) &= p_{\text{inc}} + p_{\text{ref}} - p_{\text{tr}} \\ \implies 1 + \mathcal{R} &= \left(1 - \frac{i\omega\cos\theta}{\rho_0 c_0} \right) \mathcal{T}\,. \end{aligned} \tag{2.4.68}$$

The normal particle velocity on either side of the curtain must match that of the curtain's surface. we have

$$\begin{aligned} v_{\text{curtain}} = (u_{\text{inc}} + u_{\text{ref}})\cos\theta &= u_{\text{tr}}\cos\theta \\ \implies \frac{1}{\rho_0 c_0}(p_{\text{inc}} - p_{\text{ref}}) &= \frac{1}{\rho_0 c_0} p_{\text{tr}} \\ \implies 1 - \mathcal{R} &= \mathcal{T}\,. \end{aligned} \tag{2.4.69}$$

Combining Eqs. (2.4.68) and (2.4.69) allows us to solve for the reflection and

transmission coefficients

$$\boxed{\mathcal{R} = \frac{i\omega m \cos\theta/2\rho_0 c_0}{i\omega m \cos\theta/2\rho_0 c_0 - 1}} \tag{2.4.70}$$

$$\boxed{\mathcal{T} = \frac{1}{1 - i\omega m \cos\theta/2\rho_0 c_0}}\,. \tag{2.4.71}$$

Stiff Thin Partition Consider a plane wave obliquely incident on a thin ($kd \ll 1$), stiff plate. We'll assume the fluid on either side of the plate is the same with sound speed c_0 and impedance Z_0. Setting up the force balance for this plate gives[†]

$$m_{\mathrm{pl}}\ddot{\xi}\left[1 - (c_{\mathrm{pl}}/c_{\mathrm{trace}})^4\right] = p_{\mathrm{inc}} + p_{\mathrm{ref}} - p_{\mathrm{tr}}\,, \tag{2.4.72}$$

where c_{pl} is the flexural wave speed in the plate, m_{pl} is its acoustic mass (i.e., mass per unit area), and $c_{\mathrm{trace}} = c_0/\sin\theta_{\mathrm{inc}}$ is the apparent speed along the panel of the phase fronts. Note that if the wave is normally incident, we have have effectively an infinite trace speed. Then, matching the normal velocity at the interface, we have

$$\dot{\xi} = \boldsymbol{u}_{\mathrm{tr}} \cdot \boldsymbol{e}_n = u_{\mathrm{tr}} \cos\theta = \frac{p_{\mathrm{tr}} \cos\theta}{Z_0}\,. \tag{2.4.73}$$

Since the excitation is harmonic, we can take the derivative in Eq. (2.4.72) and use Eq. (2.4.73) to write

$$\begin{aligned} \frac{-i\omega m \cos\theta}{Z_0}\left[1 - (v/c_{\mathrm{trace}})^4\right] p_{\mathrm{tr}} &= p_{\mathrm{inc}} + p_{\mathrm{ref}} - p_{\mathrm{tr}} \\ \left\{1 - \frac{\omega m \cos\theta}{Z_0}\left[1 - (v/c_{\mathrm{trace}})^4\right]\right\} \mathcal{T} &= 1 + \mathcal{R}\,. \end{aligned} \tag{2.4.74}$$

Our normal velocity condition is the same as the oblique incidence case

$$\begin{aligned} (\boldsymbol{u}_{\mathrm{inc}} + \boldsymbol{u}_{\mathrm{ref}}) \cdot \boldsymbol{e}_n &= \boldsymbol{u}_{\mathrm{tr}} \cdot \boldsymbol{e}_n \\ \frac{p_{\mathrm{inc}}}{Z_0}\cos\theta - \frac{p_{\mathrm{ref}}}{Z_0}\cos\theta &= \frac{p_{\mathrm{tr}}}{Z_0}\cos\theta \\ \implies 1 - \mathcal{R} &= \mathcal{T}\,. \end{aligned} \tag{2.4.75}$$

[†]The arrival at Eq. (2.4.72) isn't the point here, but a good overview is given on pp. 203–205 of Ref. 4.

Eliminating $\mathcal{R}$ in Eqs. (2.4.74) and (2.4.75)

$$\mathcal{T} = \frac{1}{1 - \left(i\omega m \cos\theta / 2Z_0\right)\left[1 - \left(c_{\mathrm{pl}}/c_0\right)\sin\theta\right]} \,. \tag{2.4.76}$$

Equation (2.4.76) implies that there is an angle $\theta = \arcsin c_0/c_{\mathrm{pl}}$ (provided $c_{\mathrm{pl}} > c_0$) such that perfect transmission is achieved. This effect is called *coincidence*. Typically, the plate wave speed c_{pl} is frequency-dependent, so this value will change with both frequency and incident angle.

Porous Absorber Suppose we have some thin, effectively rigid panel with a series of holes in it. We'll assume that these holes induce some turbulance in the flow through them in a way that doesn't depend on frequency; rather they just present some flow resistance R_f. We can imagine that the velocity condition is still satisfied, as the air can pass through the pores:

$$\begin{aligned} (\boldsymbol{u}_{\mathrm{inc}} + \boldsymbol{u}_{\mathrm{ref}}) \cdot \boldsymbol{e}_n &= \boldsymbol{u}_{\mathrm{tr}} \cdot \boldsymbol{e}_n \\ \implies 1 - \mathcal{R} &= \mathcal{T} \,. \end{aligned} \tag{2.4.77}$$

However, there will be a flow resistance in the pores such that the velocity is proportional to the pressure differential

$$\begin{aligned} p_{\mathrm{inc}} + p_{\mathrm{ref}} - p_{\mathrm{tr}} &= R_f u \\ \implies 1 + \mathcal{R} &= \left(1 + \frac{R_f}{Z_0}\right)\mathcal{T} \,, \end{aligned} \tag{2.4.78}$$

since $u = p_{\mathrm{tr}}/Z_0$.

2.5 Radiation

2.5.1 Spherical Sources

Transient Solution

Suppose a sound wave is generated by a radially-pulsating sphere, whose radius obeys

$$\begin{aligned} R(t) &= a + \xi(t) \\ \implies \dot{R} &= \dot{\xi} \,. \end{aligned} \tag{2.5.1}$$

The volume of air pushed through the surface of a sphere of radius a per unit time is

$$4\pi a^2 \dot{\xi} \equiv Q \,, \tag{2.5.2}$$

where Q is called the *volume velocity*. The momentum equation in spherical coordinates is

$$\rho_0 \frac{\partial u_r}{\partial t} = -\frac{\partial P}{\partial r} \,. \tag{2.5.3}$$

As we saw in Sec. 2.2.4, spherically symmetric solutions to the wave equation have the form

$$p(\boldsymbol{r}, t) = \frac{f(t - r/c)}{r}, \tag{2.5.4}$$

so that

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} &= \frac{1}{\rho_0} \nabla p \\ \frac{\partial u_r}{\partial t} &= \frac{1}{\rho_0} \frac{\partial}{\partial r} \left[\frac{f(t - r/c)}{r} \right] \quad \text{(spherical symmetry)} \\ &= \frac{1}{\rho_0} \left(\frac{1}{r} \frac{\partial f}{\partial r} - \frac{f}{r^2} \right) . \end{aligned} \tag{2.5.5}$$

Substitution of Eqs. (2.5.2) and (2.5.5) into Eq. (2.5.3) gives

$$\frac{1}{r} \left(\frac{\partial f}{\partial r} - \frac{f}{r} \right) = -\rho_0 \frac{\partial Q}{\partial t} \tag{2.5.6}$$

Consider what happens in Eq. (2.5.6) if $ka \ll 1$. In this case, $|f/r| \gg |\mathrm{d}f/\mathrm{d}r|$ at $r = a$, allowing us to write the result for a *simple source*[7]

$$f \simeq \frac{\rho_0}{4\pi} \dot{Q} \quad (r = a) \,. \tag{2.5.7}$$

The pressure at some distance r is then

$$p(r, t) \simeq \frac{\rho_0}{4\pi r} Q'(t - r/c), \tag{2.5.8}$$

where the prime indicates differentiation with respect to the argument. Now in Eq. (2.5.4) is a solution to the linearized wave equation, so this expression is valid for small particle velocities, i.e., that $\xi/a \ll 1$.

Harmonic Solution

If the source is harmonic ($\propto e^{-i\omega t}$), the derivative gives

$$p(r, t) \simeq \frac{-i\omega\rho_0}{4\pi r} Q_0 e^{i(kr - \omega t)}, \tag{2.5.9}$$

2.5.2 Multipoles

A technique for finding the radiation pattern for more complicated sources in the far field ($kr \gg 1$) is to model the source as a distribution of point sources. We know that in the free-field, the harmonic pressure source must obey the nonhomogeneous Helmholtz equation

$$(\nabla^2 + k^2)\tilde{p} = -\tilde{Q}(\boldsymbol{r}, \omega), \tag{2.5.10}$$

where $\tilde{Q}(\boldsymbol{r}, \omega)$ is the source strength. Consider a point source at $\boldsymbol{r}'$. From Eq. (2.5.10), we find

$$\begin{aligned}(\nabla^2 + k^2)\tilde{G} &= -4\pi\delta(\boldsymbol{r} - \boldsymbol{r}') \\ \implies \tilde{G} &= \frac{e^{ik\mathscr{r}}}{4\pi\mathscr{r}},\end{aligned} \tag{2.5.11}$$

where $\mathscr{r} = |\boldsymbol{\mathscr{r}}| = |\boldsymbol{r} - \boldsymbol{r}'|$. The function $\tilde{G}(\boldsymbol{\mathscr{r}}, \omega)$ is called the Green's function. Once we know what the field due to a point source will look like, we can consider more complicated sources by breaking them up into small elements and treating each as an individual point source. This is the approach for evaluating the field radiated by a circular piston in Sec. 2.5.4.

The velocity potential can be found from

$$\phi(\boldsymbol{r}, \omega) = -\frac{1}{4\pi}\int_{V'} \tilde{q}(\boldsymbol{r}')\,\frac{e^{-ik\mathscr{r}}}{\mathscr{r}}\,\mathrm{d}V', \tag{2.5.12}$$

where $\tilde{q}$ is the source strength density. Now in the far field, we contend that the amplitude differences are negligible. That is, we write $\mathscr{r} \simeq r$ in the denominator. Since the phase may vary rapidly in space, we must retain this phase difference in the exponent. However, we can expand this phase with a Taylor expansion to make it a bit more approachable. Consider the geometry in Fig. 2.10.

From the diagram we have that

$$\mathscr{r} \simeq r - \boldsymbol{r}' \cdot \boldsymbol{e}_r = r - R, \tag{2.5.13}$$

where $\boldsymbol{e}_r = \boldsymbol{r}/r$. Since $R \ll r$, we can expand as

$$\begin{aligned}e^{ax} &\simeq 1 + ax + \frac{1}{2}a^2x^2 + \ldots \\ \implies e^{-ikR} &\simeq 1 - ikR - \frac{1}{2}k^2R^2 + \ldots\end{aligned} \tag{2.5.14}$$

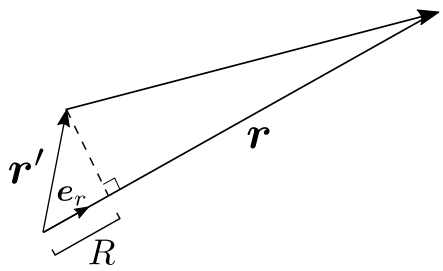


Figure 2.10: Approximation of phase difference in the far field.

The far field version of Eq. (2.5.33) becomes (note that $p = \rho_0\phi = -i\omega\rho_0\phi$)

$$p(r,\omega) \simeq -\frac{i\omega\rho_0}{4\pi r} e^{i(kr-\omega t)} \int_{V'} \left[1 - ikR - \frac{1}{2}k^2R^2\right] \tilde{q}(\boldsymbol{r}')\,\mathrm{d}V' \,. \tag{2.5.15}$$

A bit more intuitive (to me anyway) is to recognize that the source strength density integrated over the volume is really the total volume velocity of the radiating body. Typically, we have a radiating surface, and so we need to integrate the volume flow out of the surface

$$p(r,\omega) \simeq -\frac{i\omega\rho_0}{4\pi r} e^{i(kr-\omega t)} \int_{S'} \left[1 - ikR - \frac{1}{2}k^2R^2\right] \boldsymbol{u}(\boldsymbol{r}') \cdot \boldsymbol{n}\,\mathrm{d}S' \,. \tag{2.5.16}$$

The first three terms in Eq. (2.5.16) are termed the monopole, dipole, and quadrupole terms, respectively. To see why, first construct the object as a set of point sources. Then, evaluate Eq. (2.5.16) for these distributions, and see that only the corresponding term survives.

Monopoles

If we rewrite Eq. (2.5.9) by absorbing the constants into a single amplitude A, we obtain the expression for a (time harmonic) monopole

$$p = \frac{A}{r} e^{ikr} \,. \tag{2.5.17}$$

This describes a source that is infinitely compact and spherically symmetric. Of course this expression blows up as $r \to 0$, but this expression is a reasonably good approximation for acoustically compact (that is $ka \ll 1$) sources.

If we *were* to have an infinitely compact monopole source, though, we'd write

$$\boldsymbol{u}(\boldsymbol{r}') = u_0\,\delta(\boldsymbol{r}')\;\boldsymbol{e}_r \,, \tag{2.5.18}$$

so that the normal velocity is 0 except when $R = 0$. Then, only the first term of the integrand survives in Eq. (2.5.16). Now we can imagine a vanishingly small sphere of radius ϵ around the origin. Integrating over this surface, and using the sifting property of the delta function, we have

$$p = -\frac{i\omega\rho_0}{4\pi r}\left(4\pi u_0 \epsilon^2\right) e^{i(kr-\omega t)}. \tag{2.5.19}$$

Of course as we let $\epsilon \to 0$, this expression should vanish. To get around this, it is argued that the product $u_0\epsilon^2$ remain finite. This quantity has units of cubic meters per second—so it's the volume velocity! The 4π could be excluded to cancel that in the denominator, but we'll include it to define

$$\begin{aligned} Q &\equiv 4\pi u_0 \epsilon^2 \\ \implies p &= -\frac{i\omega\rho_0}{4\pi r} Q\, e^{i(kr-\omega t)}, \end{aligned} \tag{2.5.20}$$

which is Eq. (2.5.9), just as we should expect.

Dipoles

A dipole can be modeled by imagining two monopoles that are out of phase placed close together. Put the first source at the origin with source strength A. Then place the second source at $x = -d$ and give it source strength $-A$.[†] Then the total field would be

$$\begin{aligned} p_{\text{tot}} = p_1 + p_2 &= \frac{A_1}{r}e^{ikr} + \frac{A_2}{r_2}e^{ikr_2} \\ &= \frac{A}{r}e^{ikr} - \frac{A}{r_2}e^{ikr_2}. \end{aligned} \tag{2.5.21}$$

Now consider Eq. (2.5.16), with our surface drawn as a small sphere around each source as in Fig. 2.11. Since the sources are totally out of phase, the volume passing in through S_1 is equal and opposite that passing out through S_2 (and the other way around). Therefore, the first term of the integrand in Eq. (2.5.16) will vanish by symmetry.

The second term however is proportional $-ikR$. Note that for S_1 and S_2, the respective Rs will be opposite. Thus the contributions from the two sources due to this term will *add*. Because only this term survives the integration, it is called the dipole moment. The third term is proportional to $(kR)^2$, and thus

[†]This accounts for the phase difference, since $Ae^{-i(\omega t+\pi)} = Ae^{-i\pi}e^{-i\omega t} = -Ae^{i\omega t}$.

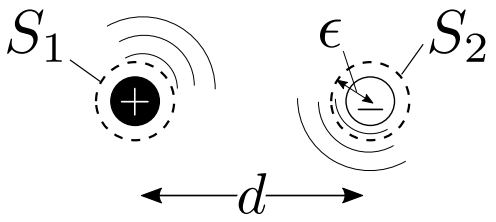


Figure 2.11: Two simple sources (monopoles) separated by d with phase difference π (i.e., totally out of phase). A with a small spherical surface of radius ϵ is drawn around each.

the sign of R is not preserved, and it, like the monopole term, vanishes by symmetry.

If consider the far-field pattern radiated by our two-point dipole, this should approximate the field of a true dipole, since as we let $d \to 0$, for any finite radius r, $d/r \ll 1$. In this case, we claim that we can take the factor r in the denominator of each monopole term to be about the same. This is because the difference between the distance to each source differs only by d, and our far field assumption is that $d + r \simeq r$, so that $r_2^{-1} \simeq r^{-1}$. We cannot make this claim about the phase, so we are left with

$$p_{\text{tot}} = \frac{A}{r} e^{ikr} \left[1 - e^{ik(r_2 - r)}\right] . \tag{2.5.22}$$

Now from the geometry, we can see that (since kr is large)

$$r_2 - r \simeq d \cos\theta , \tag{2.5.23}$$

and so Eq. (2.5.22) becomes

$$p_{\text{tot}} = \frac{A}{r} e^{ikr} \left(1 - e^{ikd\cos\theta}\right) . \tag{2.5.24}$$

Our last step is to enforce the requirement that kd is small. In this case, since $|\cos\theta| \sim 1$, we perform a Taylor expansion of the exponential term and write

$$p_{\text{tot}} = \frac{A}{r} e^{ikr} \left(-ikd\cos\theta\right) . \tag{2.5.25}$$

Quadrupoles

Analysis of the quadrupole would follow in a similar manner, except now we could arrange our sources in two ways, either **(1)** as two dipoles, with

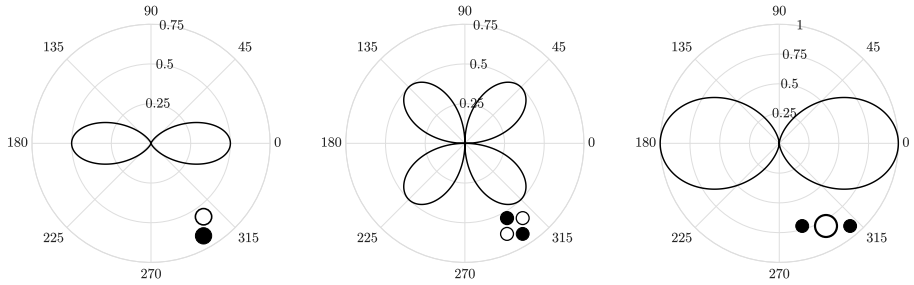


Figure 2.12: Directivity patterns of a dipole (left), and lateral (center) & linear (right) quadrupoles with $d = a$. The simple (monopole) sources are arranged as shown, with solid circles having opposite phase as hollow circles.

opposite orientation and separated by distance a placed side-by-side (lateral quadrupole); or **(2)** as two dipoles, with opposite orientation and separated by distance a along a single one axis (linear quadrupole). The analysis is more complicated and I don't want to do it out, so we'll take the experts' word for it[8] that the fields radiated are

$$p_{\text{tot}} = \frac{A}{r} e^{ikr} \left(4k^2 da \cos\theta \sin\theta\right) \qquad \text{(lateral)} \tag{2.5.26}$$

$$p_{\text{tot}} = \frac{A}{r} e^{ikr} \left(4k^2 da \cos^2\theta\right) \qquad \text{(linear)}\,. \tag{2.5.27}$$

Qualitative for multipole radiation patterns are shown in Fig. 2.12.

2.5.3 Solution Techniques

This section address the problem of finding the field radiated by a general source distribution. This is the external problem. The internal problem is one of waveguides, which should probably be in here somewhere (but currently aren't).

Kirchhoff-Helmholtz Integral

Consider the vector identity[1]

$$\nabla \cdot (f\nabla g - g\nabla f) = f\left(\nabla^2 + a^2\right) g - g\left(\nabla^2 + a^2\right) f\,, \tag{2.5.28}$$

where f and g are arbitrary functions of space and a is a constant (which may be 0, in which case this identity is more immediately evident). Now for some

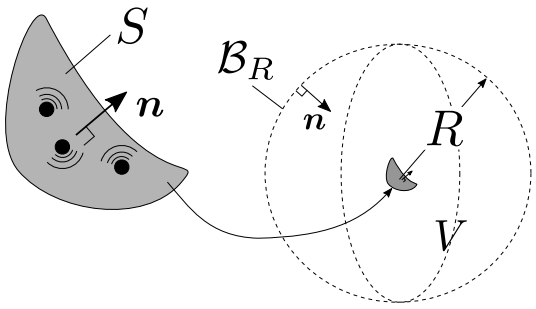


Figure 2.13: The surface S bounds all sources, while V is the region exterior to S and within a large sphere of radius R.

clever manipulations.

For reasons that will become clearer, let f be the Green's function given by Eq. (2.5.11), let g be the harmonic pressure field p, and let our arbitrary constant be the wavenumber $k = \omega/c_0$. And suppose all of our (harmonic, but otherwise arbitrary) sources are confined within some region bounded by a surface S. Then, let's integrate both sides over the volume contained between S and a large sphere of radius R (see Fig. 2.13).

$$\underbrace{\int_V \nabla\cdot\left(\tilde{G}\nabla p - p\nabla\tilde{G}\right)\,\mathrm{d}V}_{\text{I}} = \underbrace{\int_V \tilde{G}\left(\nabla^2+k^2\right)p\,\mathrm{d}V}_{\text{II}} - \underbrace{\int_V p\left(\nabla^2+k^2\right)\tilde{G}\,\mathrm{d}V}_{\text{III}}\,. \tag{2.5.29}$$

Immediately, we can say that term II of Eq. (2.5.29) vanishes, since there are no sources in V, and thus $(\nabla^2 + k^2)p = 0$. For the term III, we'll use our form of the Greens function from Eq. (2.5.11)

$$\begin{aligned}\int_V \tilde{G}\left(\nabla^2+k^2\right)p\,\mathrm{d}V &= \int_V p\left(\nabla^2+k^2\right)\left[-4\pi\delta(\boldsymbol{r})\right]\,\mathrm{d}V\\ &= -4\pi p(\boldsymbol{r}),\end{aligned} \tag{2.5.30}$$

where $\boldsymbol{r}'$ lies on the surface S. Finally, we'll use the divergence theorem to

write the term I of Eq. (2.5.29) as

$$\begin{aligned}\int_V \nabla\cdot\left(\tilde{G}\nabla p - p\nabla\tilde{G}\right)\,\mathrm{d}V &= \int_{S+\mathcal{B}_R}\left(\tilde{G}\nabla p - p\nabla\tilde{G}\right)\cdot\boldsymbol{n}\,\mathrm{d}S\\ &= -\int_S\left(\tilde{G}\nabla p - p\nabla\tilde{G}\right)\cdot\boldsymbol{n}\,\mathrm{d}S\\ &\quad+\int_{\mathcal{B}_R}\left(\tilde{G}\nabla p - p\nabla\tilde{G}\right)\cdot\boldsymbol{n}\,\mathrm{d}S\,. \qquad (2.5.31)\end{aligned}$$

Note that the signs of the terms the right of Eq. (2.5.31) are opposite, since their normal vectors are opposed. We've chosen the negative orientation (unit vectors oriented *inward*) for convenience. However we can get rid of this last term altogether if we argue that $\tilde{G}$ and p are small on the outer surface $\mathcal{B}_R$. This should be the case, as long as R is large—and we haven't yet made any assumptions about R—so we can safely neglect the contribution on the outer surface.[†] The gradient of a function dotted with the normal vector (i.e., the directional derivative) is often called the normal derivative, written

$$\nabla f\cdot\boldsymbol{n} = \frac{\partial f}{\partial n}\,. \qquad (2.5.32)$$

So, with all these simplifications, and using the Green's function definition, we can write Eq. (2.5.29) as

$$\begin{aligned}-4\pi p(\boldsymbol{r}) &= -\int_S\left(\frac{e^{ik\mathscr{r}}}{\mathscr{r}}\,\frac{\partial p}{\partial n} - p\frac{\partial\tilde{G}}{\partial n}\right)\mathrm{d}S\,, \qquad \text{or}\\ p(\boldsymbol{r}) &= \frac{1}{4\pi}\int_S\left(\frac{e^{ik\mathscr{r}}}{\mathscr{r}}\,\frac{\partial p}{\partial n} - p\frac{\partial\tilde{G}}{\partial n}\right)\mathrm{d}S\,. \qquad (2.5.33)\end{aligned}$$

Equation (2.5.33) is the *Kirchoff-Helmholtz Integral*, and states that we can find the pressure field external to a surface which bounds an arbitrary distribution of sources, provided that we know the pressure and its normal derivative on that surface. Usually we use a modifi

Rayleigh Integral

The Rayleigh integral is a special case of Eq. (2.5.33), where we use the fact that

$$\frac{\partial p}{\partial n} = \nabla p\cdot\boldsymbol{n} = \rho_0\frac{\partial\boldsymbol{u}}{\partial t}\cdot\boldsymbol{n} \qquad (2.5.34)$$

[†]More precisely, this statement invokes the Sommerfield radiation condition, which requires that $\tilde{G}$ and p must decay at least as fast as r^{-1}. This is true for just the geometric spreading of both, but will be even more comfortably satisfied if we include any sort of attenuation (which would exist in any real propagation).

It gives an expression for the pressure due to an arbitrary source as a summation of the contributions from the source

$$p(\boldsymbol{r},t) = \frac{\rho_0}{4\pi}\int_S \frac{e^{ik\mathscr{r}}}{\mathscr{r}}\dot{\boldsymbol{u}}(\boldsymbol{r}')\cdot\boldsymbol{e}_n\,\mathrm{d}S\,. \tag{2.5.35}$$

The primed coordinates refer to coordinates on the source surface, and the distance $\mathscr{r} = |\boldsymbol{\mathscr{r}}|$, where

$$\boldsymbol{\mathscr{r}} \equiv \boldsymbol{r} - \boldsymbol{r}'\,. \tag{2.5.36}$$

2.5.4 The Baffled Piston

Consider a piston surrounded by a rigid infinite baffle. We could model this arrangement as a surface whose normal velocity is 0 everywhere except within the area of the piston face, where it has some normal velocity given by $\boldsymbol{u} = u(\boldsymbol{r}')\boldsymbol{e}_z$.[†] We could compute the pressure at any point using the Rayleigh integral with one slight modification: the denominator of the source term becomes 2π rather than 4π. From Ref. 4:

> “*[The Rayleigh integral] is derived on the assumption that the simple sources on the distributed area radiate into full space, whereas in the case of the baffled piston, the baffle restricts the radiation to the forward hemisphere. Consequently, the factor of* 4π *in the denominator of [Eq.* (2.5.35)*] must be replaced by* 2π*. Another way to justify the change to* 2π *is to recognize that restricting radiation to a half space effectively doubles the strength of the simple sources.*”
>
> —p. 441 of Ref. 4

Then,

$$\begin{aligned} p(\boldsymbol{r},t) &= \rho_0\int_S \frac{e^{ik\mathscr{r}}}{2\pi\mathscr{r}}\,\dot{\boldsymbol{u}}(\boldsymbol{r}')\cdot\boldsymbol{e}_n\,\mathrm{d}A' \\ &= \rho_0\int_S \frac{e^{ik\mathscr{r}}}{2\pi\mathscr{r}}\,\dot{u}(\boldsymbol{r}')\,\mathrm{d}A', \end{aligned} \tag{2.5.37}$$

[†]Note that this approximation means we are modeling a surface that *does not move* out of the xy-plane, yet somehow manages to have some velocity $u(\boldsymbol{r}')$. The implication is then that this is model is valid only if the displacements of the piston are small compared to a wavelength.

If we further assume that the piston velocity is time-harmonic and uniform across the piston face, we have

$$u(\boldsymbol{r}') = u_0 e^{-i\omega t} \implies \dot{u} = -i\omega u_0 e^{-i\omega t} \tag{2.5.38}$$

so that Eq. (2.5.37) becomes

$$\begin{aligned} p(\boldsymbol{r}, t) &= \rho_0 \int_S \frac{e^{ik\mathscr{r}}}{2\pi\mathscr{r}} - i\omega u_0 e^{-i\omega t} \, \mathrm{d}A' \\ &= -\frac{i\omega\rho_0 u_0 e^{-i\omega t}}{2\pi} \int_S \frac{e^{ik\mathscr{r}}}{\mathscr{r}} \, \mathrm{d}A' \\ &= -\frac{ik\rho_0 c_0 u_0 e^{-i\omega t}}{2\pi} \int_S \frac{e^{ik\mathscr{r}}}{\mathscr{r}} \, \mathrm{d}A' \,. \end{aligned} \tag{2.5.39}$$

The presence of $\mathscr{r}$ in Eq. (2.5.39) makes it generally hard to work with. But we can look at some special cases that make the integral a bit more tractable.

Circular Piston: On-axis Pressure

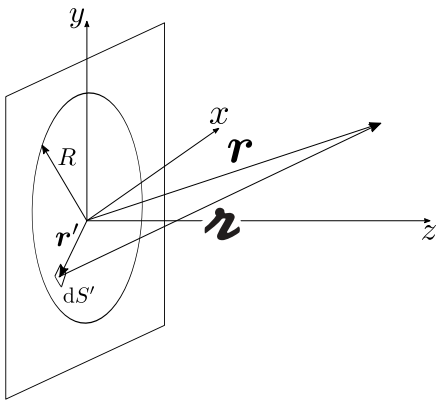


Figure 2.14: Circular Piston Geometry

If we consider points along the z axis for a circular piston of radius R, then we have the advantage the problem is now axially symmetric and the dimension of the problem is reduced; see Fig. 2.14. Then, changing to spherical coordinates,

we have

$$\begin{aligned} p(z,t) &= -\frac{ik\rho_0 c_0 u_0 e^{-i\omega t}}{2\pi} \int_0^{2\pi}\int_0^R \frac{e^{ik\imath}}{\imath} r'\,\mathrm{d}r'\,\mathrm{d}\phi' \\ &= -\frac{ik\rho_0 c_0 u_0 e^{-i\omega t}}{2\pi} \underbrace{\int_0^{2\pi} \mathrm{d}\phi'}_{=2\pi} \int_0^R \frac{r' e^{ik\imath}}{\imath}\,\mathrm{d}r' \\ &= -ik\rho_0 c_0 u_0 e^{-i\omega t} \int_0^R \frac{r' e^{ik\imath}}{\imath}\,\mathrm{d}r' . \end{aligned} \tag{2.5.40}$$

Now in this case

$$\imath = |\boldsymbol{\imath}| = \sqrt{z^2 + r'^2} \tag{2.5.41}$$

so that the integrand becomes

$$\frac{r' \exp ik\imath}{\imath} = \frac{r' \exp ik\sqrt{z^2 + r'^2}}{\sqrt{z^2 + r'^2}} . \tag{2.5.42}$$

But notice that

$$\frac{\mathrm{d}}{\mathrm{d}r'}\left(\frac{\exp ik\sqrt{z^2 + r'^2}}{ik}\right) = \frac{r' \exp ik\sqrt{z^2 + r'^2}}{\sqrt{z^2 + r'^2}}, \tag{2.5.43}$$

so from Eqs. (2.5.40) and (2.5.43) our on-axis pressure is

$$\begin{aligned} p(z,t) &= -ik\rho_0 c_0 u_0 e^{-i\omega t}\left[\frac{\exp ik\sqrt{z^2 + r'^2}}{ik}\Bigg|_0^R\right] \\ &= \rho_0 c_0 u_0 \left(e^{ikz} - e^{ik\sqrt{z^2+R^2}}\right) e^{-i\omega t}. \end{aligned} \tag{2.5.44}$$

As Ref. 4 ponts out, Eq. (2.5.44) has the form of two plane waves; one emanating from distance z, and one from the edge of the piston at distance $\sqrt{z^2 + R^2}$.

Circular Piston: Far-field Pressure

Suppose now that we're allowed to go away from the z axis, but that we are relatively far from the piston (specifically, $R/r \ll 1$). The argument is that the differences in path lengths $\imath$ over the face of the transducer are important only in the phase difference. That is,

$$\frac{e^{ik\imath}}{\imath} \simeq \frac{e^{ik\imath}}{r}. \tag{2.5.45}$$

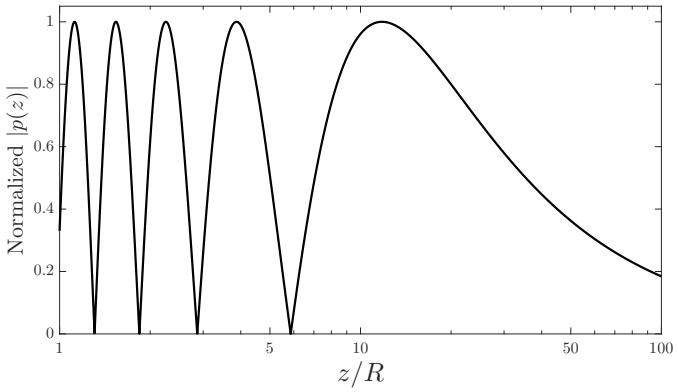


Figure 2.15: Pressure along the z axis due to a circular piston of radius R. Note that here $kR \simeq 75$. The Rayleigh distance is proportional to frequency, so the onset of the far-field moves closer at low frequencies and further away at high frequencies.

Integrating over the surface of the piston S is most naturally done in polar coordinates. Let the field point $\boldsymbol{r}$ be in the y-z plane.[†] Let θ be the angle between $\boldsymbol{r}$ and the z axis, so that

$$\boldsymbol{r} = r\sin\theta\boldsymbol{e}_y + r\cos\theta\boldsymbol{e}_z\,. \tag{2.5.46}$$

The surface elements are located at

$$\boldsymbol{r}' = r'\sin\phi\boldsymbol{e}_y + r'\cos\phi\boldsymbol{e}_x\,. \tag{2.5.47}$$

Then,

$$\begin{aligned}\mathscr{r}^2 &= r'^2\cos^2\phi + (r\sin\theta - r'\sin\phi)^2 + r^2\cos^2\theta\\ &= r'^2\cos^2\phi + r^2\sin\theta^2 - 2rr'\sin\theta\sin\phi + r'^2\sin^2\phi + r^2\cos^2\theta\\ &= r'^2 + r^2 - 2rr'\sin\theta\sin\phi\,.\end{aligned} \tag{2.5.48}$$

Since we've required $R/r \ll 1$, and since $r' < R$, we can expand Eq. (2.5.48) to find

$$\begin{aligned}\mathscr{r} &= \sqrt{r'^2 + r^2 - 2rr'\sin\theta\sin\phi}\\ &= r\left[1 + (r'/r)^2 - 2(r'/r)\sin\theta\sin\phi\right]^{\frac{1}{2}}\\ &\simeq r\left[1 - (r'/r)\sin\theta\sin\phi + \mathcal{O}(r'/r)^2\right]\\ &\simeq r - r'\sin\theta\sin\phi\,.\end{aligned} \tag{2.5.49}$$

[†]This assumption is made without loss of generality; the position of the x and y axes is arbitrary since the problem is symmetric.

The expression for the far-field pressure is then

$$p(\boldsymbol{r},t) = -\frac{ik\rho_0 c_0 u_0 e^{-i\omega t}}{2\pi}\int_0^{2\pi}\int_0^R \frac{e^{ik(r-r'\sin\theta\sin\phi)}}{r}\, r'\,\mathrm{d}r'\,\mathrm{d}\phi$$
$$= -\frac{ik\rho_0 c_0 u_0 e^{-i(\omega t-kr)}}{2\pi r}\int_0^{2\pi}\int_0^R e^{-i(kr'\sin\theta)\sin\phi}\, r'\,\mathrm{d}r'\,\mathrm{d}\phi\,. \qquad (2.5.50)$$

Now, we note that the integral should be symmetric so that we can change the ϕ limits to $[0,\pi]$ and double the result. Because smarter people than I have already worked this problem, they suggest we recall that the Bessel functions may be generated from

$$J_n(x) = \frac{-i^{-m}}{\pi}\int_0^\pi e^{-ikx\sin\theta}\cos m\theta\,\mathrm{d}\theta\,. \qquad (2.5.51)$$

Then, we recognize that Eq. (2.5.50) simplifies to

$$p(\boldsymbol{r},t) = -\frac{ik\rho_0 c_0 u_0 e^{-i(\omega t-kr)}}{\pi r}\int_0^R J_0(kr'\sin\theta)\, r'\,\mathrm{d}r' \qquad (2.5.52)$$

(the factor of 2 in the denominator was canceled by the symmetry of the integral in ϕ). Finally, set $w = kr'\sin\theta$ and then $r'\mathrm{d}r' = w\,\mathrm{d}w/(k\sin\theta)^2$, so that

$$\int_0^R J_0(kr'\sin\theta)\, r'\,\mathrm{d}r' = \frac{1}{k^2\sin^2\theta}\int_0^{kR\sin\theta} w\,J_0(w)\,\mathrm{d}w$$
$$= \frac{R}{k\sin\theta}J_1(kR\sin\theta)\,. \qquad (2.5.53)$$

And so, our final expression for the far-field pressure radiated by the piston is

$$\boxed{p(\boldsymbol{r},\theta,t) = -\frac{i\rho_0 c_0 u_0 R}{\pi r\sin\theta}\,J_1(kR\sin\theta)\,e^{i(kr-\omega t)}} \qquad (2.5.54)$$

Equation (2.5.54) is sometimes rearranged and written

$$p(\boldsymbol{r},\theta,t) = A\frac{R}{r}D(\theta)\,e^{i(kr-\omega t)}, \qquad (2.5.55)$$

where

$$D(\theta) = \frac{J_1\left(kR\sin\theta\right)}{kR\sin\theta} \qquad (2.5.56)$$

is the directivity of the source. Plots of this directivity functions for a few values of kR are shown in Fig. 2.16. At low kR (Fig. 2.16a) we see the source becomes omnidirectional, whereas at higher kR (Fig. 2.16c), the main lobe is much narrower, and side lobes begin to appear.

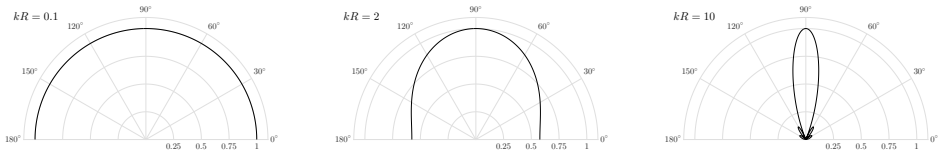


Figure 2.16: Plots of the directivity of the field radiated by a cicular piston for **(a)** $kR = 0.1$, **(b)** $kR = 2$, and **(c)** $kR = 10$.

Circular Piston: Impedance

The radiation impedance is not trivial to derive. The main idea is that we must calculate the force on each surface element of the piston due to the pressure radiated by every other surface element of the piston. By noting that these forces must be the same, and using some recurrence relations, it can be shown that

$$Z_{\text{rad}} = \rho_0 c_0 \left[R_1(2ka) - iX_1(2ka) \right], \tag{2.5.57}$$

where R_1 and X_1 are the piston functions. They are defined

$$R_1(\xi) = 1 - 2J_1(\xi)/\xi, \quad \text{and} \tag{2.5.58}$$

$$X_1(\xi) = 2K_1(\xi)/\xi, \tag{2.5.59}$$

where $K_1(\xi)$ is the Struve function of order 1. These functions are plotted in Fig. 2.17. I can't imagine we'd be asked to derive or even remember anything but Eq. (2.5.57).

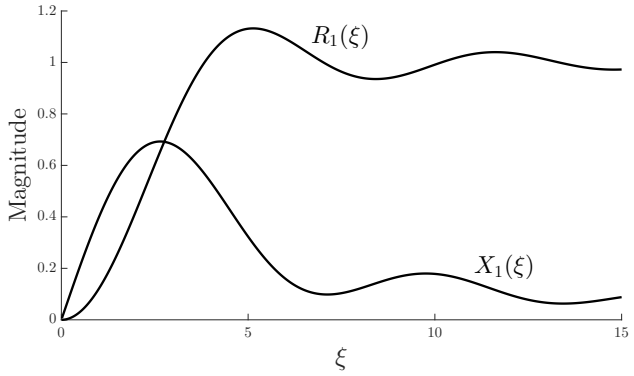


Figure 2.17: Plot of the piston resistance $R_1(\xi)$ [Eq. (2.5.58)] and reactance $X_1(\xi)$ [Eq. (2.5.59)] functions. Note that the argument of these functions is $2ka$.

NB For some reason, it is common practice to call by "radiation impedance" the *mechanical* impedance seen by the piston

$$Z_{\text{rad}} = \pi a^2 \rho_0 c_0 \left[R_1(2ka) - i X_1(2ka) \right] . \quad \text{(Mechanical)} \tag{2.5.60}$$

References 4 and 9 do the sensible thing and define "radiation impedance" with Eq. (2.5.57)—which is an acoustic impedance. The conversion is trivial (we just multiply Z_{rad} by the piston area to get the force rather than the pressure), but the terminology confusing. Rayleigh[10] speaks only of "forces" and "reactions", but most other sources[1,6,7,11] use the mechanical definition.

3

Applied Math

"Now I will have less distraction."
— Leonard Euler[†]

3.1 Stated Topics

The applied mathematics qualifying exam draws from selected topics in vector calculus, linear algebra, linear ordinary differential equations (ODEs), linear partial differential equations (PDEs), and elementary numerical analysis. The examination will be based on materials normally covered in the following courses: MATH 4305 (Topics in Linear Algebra), MATH 4581 (Classical Mathematical Methods in Engineering), and ME 2016 (Computer Applications).

[†]This was Euler's reaction to the following events:

> "*In 1735 [...] Euler received a problem in celestial mechanics from the French Academy. Though other mathematicians had required several months to solve this problem, Euler, using the improved methods of his own and by devoting intense concentration to it, solved in in three days and the better part of the two intervening nights. [...] The strain of the effort induced a fever from which Euler finally recovered, but with the loss of the sight of his right eye.*"
>
> —p. 241 of Ref. 12

I suppose we should be glad that, though arduous, studying for this exam (probably) won't cost us an eye.

The applied math exam is mostly difficult because of the breadth of material that's available to be tested. I don't really have any tricks or advice to offer other than to do all the practice exams and rote memorization that this demands. I got some advice while studying that it was fruitless to try and memorize everything, and instead it's best to shore up the things you feel you're better at. I wasn't comfortable writing off any particular topic, and your mileage may vary, but this probably isn't the worst advice.

3.2 Vector Calculus

3.2.1 Operators

Gradient

In Cartesian coordinates, the gradient operator is $\nabla = \partial_i \boldsymbol{e}_i$, so that

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{e}_x + \frac{\partial f}{\partial y}\boldsymbol{e}_y + \frac{\partial f}{\partial z}\boldsymbol{e}_z \,. \tag{3.2.1}$$

In cylindrical coordinates, this becomes

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi + \frac{\partial f}{\partial z}\boldsymbol{e}_z \,, \tag{3.2.2}$$

and in spherical coordinates

$$\nabla f = \frac{\partial f}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\boldsymbol{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\boldsymbol{e}_\phi \,. \tag{3.2.3}$$

Note that as shown in Fig. 3.1, θ is the polar angle (between the z-axis and $\boldsymbol{r}$) and ϕ is the azimuthal angle (between the x-axis and the projection of $\boldsymbol{r}$ in the xy plane). Some conventions swap the labels of θ and ϕ, so use caution when applying these formulas.

The gradient vector is normal to the surface. A unit normal to the surface $z = f(x, y)$ can be found by writing $g = z - f(x, y)$, and then

$$\boldsymbol{n} = \frac{\nabla g}{\|\nabla g\|} \,. \tag{3.2.4}$$

The tangent plane to a to g at some point $(x_0,\, y_0,\, z_0)$ is therefore given by

$$n_x(x - x_0) + n_y(y - y_0) + n_z(z - z_0) \,. \tag{3.2.5}$$

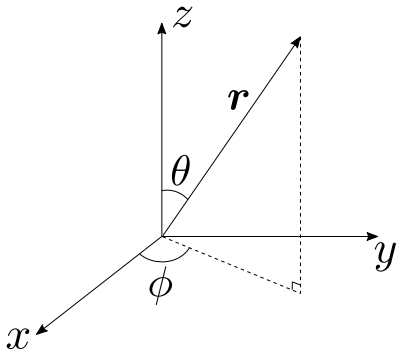


Figure 3.1: Coordinate system naming conventions used.

Divergence

The divergence of a vector field $\boldsymbol{A}$ is defined as

$$\operatorname{div} \boldsymbol{A} \equiv \lim_{\Omega \to \mathcal{P}} \frac{1}{|\Omega|} \int_{\partial\Omega} \boldsymbol{A} \cdot \boldsymbol{e}_n \, \mathrm{d}S \,. \tag{3.2.6}$$

Examination of Eq. (3.2.6) indicates that the divergence is a measure of how much the vector field "diverges" from a point $\mathcal{P}$. We expect the divergence to be maximized when the vector field points radially out from a point (i.e., $\boldsymbol{A}$ is always in line with $\boldsymbol{e}_n$).

The divergence is usually written $\nabla \cdot \boldsymbol{A}$, since it can be evaluated by performing the dot product with the operator ∇. Explicitly then for Cartesian coordianates

$$\nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \,, \tag{3.2.7}$$

where $A_i = \boldsymbol{A} \cdot \boldsymbol{e}_i$ is the component of $\boldsymbol{A}$ in the i^{th} direction. In cylindrical coordinates

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \,, \tag{3.2.8}$$

and in spherical coordinates

$$\nabla \cdot \boldsymbol{A} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 A_r \right) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta} \left(A_\theta \sin\theta \right) + \frac{1}{r \sin\theta} \frac{\partial A_\phi}{\partial \phi} \,. \tag{3.2.9}$$

Curl

While the divergence measures how much of a vector field diverges from a point, the rotation is a measure of how much the vector field rotates around a given unit vector $\hat{\boldsymbol{a}}$. The rotation of $\boldsymbol{A}$ is defined as

$$\operatorname{rot}_{\hat{a}} \boldsymbol{A} \equiv \lim_{\Omega \to \mathcal{P}} \frac{1}{|\Omega|} \int_{\partial\Omega} \boldsymbol{A} \cdot \boldsymbol{e}_t \, \mathrm{d}S \,. \tag{3.2.10}$$

The curl of a vector field is its rotation about the coordinate unit vectors; that is

$$\operatorname{curl} \boldsymbol{A} \equiv \left(\operatorname{rot}_{e_x} \boldsymbol{A}\right) \boldsymbol{e}_x + \left(\operatorname{rot}_{e_y} \boldsymbol{A}\right) \boldsymbol{e}_y + \left(\operatorname{rot}_{e_z} \boldsymbol{A}\right) \boldsymbol{e}_z \tag{3.2.11}$$

Much like the divergence, the curl of a vector field $\boldsymbol{A}$ is usually written $\nabla \times \boldsymbol{A}$, and evaluated similarly. However the cross product is a bit tedious to carry out, so only a handy mnemonic to obtain the Cartesian result is shown:

$$\nabla \times \boldsymbol{A} = \det \begin{pmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{pmatrix} \tag{3.2.12}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \boldsymbol{e}_x - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) \boldsymbol{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \boldsymbol{e}_z. \tag{3.2.13}$$

3.2.2 Integration

Double and Triple Integrals

Extension of the usual $1D$ integration to areas and volumes is relatively straightforward. Usually, the tricky part is choosing the limits for the variables correctly.

Line Integrals

To compute the integral of a function $f(x, y)$ along some $2D$ curve C, parameterize the curve and the variables so that the integral reduces to a regular 1-D integral. That is, suppose

$$\int_C f(x, y) \, \mathrm{d}s \,. \tag{3.2.14}$$

Then let $x = x(t)$, $y = y(t)$. The differential arc length can be found by considering a small change in x and y:

$$\begin{aligned}(\Delta s) &= (\Delta x)^2 + (\Delta y)^2 \\ &= \left(\frac{\mathrm{d}x}{\mathrm{d}t}\Delta t\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\Delta t\right)^2 .\end{aligned} \tag{3.2.15}$$

Allowing $\Delta t \to 0$, we have

$$\mathrm{d}s = \sqrt{(\mathrm{d}x/\mathrm{d}t)^2 + (\mathrm{d}y/\mathrm{d}t)^2}\,\mathrm{d}t\,. \tag{3.2.16}$$

We've now reduced our line integral to a regular Calc II integral. We could extend to three-dimensional curves too, by allowing $z = z(t)$ and proceeding as before.

Surface Integrals

Suppose now we want to integrate a function over a whole surface, rather than just along a curve. In this case we sum up over differential surface elements. To find the surface area element $\mathrm{d}S$ of a surface $S = \{(x, y, z) : z = f(x, y)\}$, consider the tangent vectors $\boldsymbol{t}_x$ and $\boldsymbol{t}_y$ (see Fig. 3.2). The area of the region they bound is $\|\boldsymbol{t}_x \times \boldsymbol{t}_y\|$.

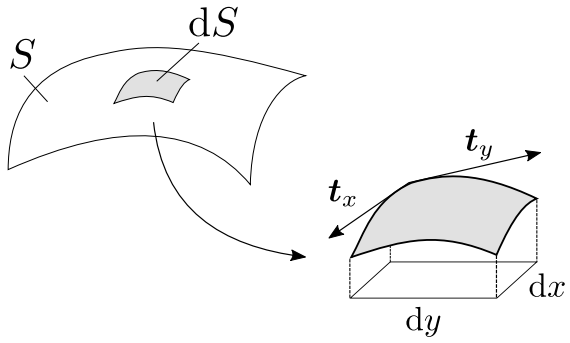


Figure 3.2: Element of surface area with unit tangent vectors $\boldsymbol{t}_x$ and $\boldsymbol{t}_y$.

Now

$$\boldsymbol{t}_x = \mathrm{d}x\,\boldsymbol{e}_x + \frac{\partial f}{\partial x}\,\mathrm{d}x\,\boldsymbol{e}_z \tag{3.2.17}$$

$$\boldsymbol{t}_y = \mathrm{d}y\,\boldsymbol{e}_y + \frac{\partial f}{\partial y}\,\mathrm{d}y\,\boldsymbol{e}_z\,. \tag{3.2.18}$$

Thus

$$\begin{aligned} \mathrm{d}S &= \|\boldsymbol{t}_x \times \boldsymbol{t}_y\| \\ &= \left\| \left(-\frac{\partial f}{\partial x}\,\mathrm{d}x\,\mathrm{d}y \right) \boldsymbol{e}_x + \left(-\frac{\partial f}{\partial y}\,\mathrm{d}y\,\mathrm{d}x \right) \boldsymbol{e}_y + (\mathrm{d}x\,\mathrm{d}y)\,\boldsymbol{e}_z \right\| \\ &= \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\,\mathrm{d}A\,. \end{aligned} \tag{3.2.19}$$

As we might expect, Eq. (3.2.19) shows that $\mathrm{d}S = \mathrm{d}A$ when the surface is flat ($f_x = f_y = 0$).

3.2.3 Important Theorems

Green's Theorem

Green's theorem allows us to write an integral of some function along a closed curve C as an integral over the area that it bounds. Suppose we have two functions P and Q, which are functions of x and y and are defined on the open region that contains some area S. If C is the curve that bounds the area S, then Green's theorem says

$$\oint_C (P\,\mathrm{d}x + Q\,\mathrm{d}y) = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\,\mathrm{d}y \tag{3.2.20}$$

Suppose we want to calculate the work done to move along $\mathcal{C}$ through an applied force field $\boldsymbol{F}$. The work is[†]

$$W = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} \tag{3.2.21}$$

Then, since $\mathrm{d}r = \mathrm{d}x\boldsymbol{e}_x + \mathrm{d}y\boldsymbol{e}_y$, we could instead write

$$\begin{aligned} W &= \oint_C (F_1\,\mathrm{d}x + F_2\,\mathrm{d}y) \\ &= \int_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \mathrm{d}A\,. \end{aligned} \tag{3.2.22}$$

Green's theorem also allows us to find a nice expression for the area of planar region by integrating along its boundary. Note that the area of a region R is

$$\text{Area} = \int_R \mathrm{d}A\,. \tag{3.2.23}$$

[†]We dot with $\mathrm{d}\boldsymbol{r}$ since work is done only in the direction of displacement.

So, if we can find some vector field $\boldsymbol{F}$ such that

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 1\,, \tag{3.2.24}$$

we can use Green's theorem to write

$$\text{Area} = \int_R (1)\,\mathrm{d}A = \int_R \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathrm{d}A = \oint_{\partial R} F_x\,\mathrm{d}x + F_y\,\mathrm{d}y\,. \tag{3.2.25}$$

By inspection, we might guess $\boldsymbol{F} = -y\,\boldsymbol{e}_x + x\,\boldsymbol{e}_y$. This gives $(\partial F_y/\partial x) - (\partial F_x/\partial y) = 2$. So dividing by 2 gives us the vector field we want

$$\boldsymbol{F} = \frac{1}{2}\left(-y\,\boldsymbol{e}_x + x\,\boldsymbol{e}_y\right)\,. \tag{3.2.26}$$

Thus our formula for area becomes

$$\text{Area} = \frac{1}{2}\oint_{\partial R} -y\,\mathrm{d}x + x\,\mathrm{d}y\,. \tag{3.2.27}$$

Stokes' Theorem

Suppose we have a surface S, which is bounded by a curve ∂S. Then Stokes' theorem says

$$\int_S (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n}\,\mathrm{d}S = \oint_{\partial S} \boldsymbol{A} \cdot\,\mathrm{d}\boldsymbol{r}\,. \tag{3.2.28}$$

Stokes' theorem is similar to Green's theorem in that it allows us to relate an integral over a surface to an integral around the curve that defines the boundary of the surface. However Stokes' theorem works for any surface, while Green's theorem requires the surface to be flat.

Divergence Theorem

The divergence theorem (also known as Gauss' theorem, but we already have too many things named after him) states that for a continuously differentiable vector field $\boldsymbol{A}$, we can write

$$\int_V \nabla \cdot \boldsymbol{A}\,\mathrm{d}V = \oint_{\partial V} \boldsymbol{A} \cdot\,\mathrm{d}\boldsymbol{s} \tag{3.2.29}$$

$$= \oint_{\partial V} \boldsymbol{A} \cdot \boldsymbol{e}_n\,\mathrm{d}S\,, \tag{3.2.30}$$

where the closed surface ∂V bounds the volume V.

Some Facts

On a few past exams, there were some items worth noting. Nothing revolutionary here, but they may come in handy and I didn't see them at first.

1. **Stokes' theorem applied to closed surface will vanish**
   This seems like it should be the case intuitively; as surface becomes closed, its boundary must get smaller and then vanish. Integrating over a curve of length 0 must of course give 0.

   More directly, we could note that if the surface is closed, then we can use the divergence theorem, i.e.,

   $$\int_{\partial S} (\nabla \times \boldsymbol{A}) \cdot \boldsymbol{n} \, \mathrm{d}S = \int_V \nabla \cdot (\nabla \times \boldsymbol{A}) \, \mathrm{d}V = 0, \tag{3.2.31}$$

   since the divergence of the curl is always 0. So if you see that a vector field has a vector potential, i.e., $\boldsymbol{F} = \nabla \times \boldsymbol{A}$, then you know immediately that its integral around any closed surface must vanish.

2. **Divergence theorem may make surface integral easier**
   Suppose we want to evaluate the surface integral

   $$\int_S \boldsymbol{A} \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{3.2.32}$$

   where S is, say, the hemisphere above the xy plane. Unless $\boldsymbol{A}$ has a nice form, the dot product over the spherical surface might get icky. If we are lucky and can find a vector potential $\boldsymbol{B}$ such that $\boldsymbol{A} = \nabla \times \boldsymbol{B}$, then we could just use Stokes' theorem to write

   $$\int_S \boldsymbol{A} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{r}, \tag{3.2.33}$$

   and $C = \partial S$ is the curve bounding the hemispherical surface S in Fig. 3.3. But this is unlikely too.

   Instead, we can note that

   $$\int_S = \int_{S+S'} - \int_{S'}. \tag{3.2.34}$$

The first term on the right of Eq. (3.2.34) is a closed surface, so we can use to divergence theorem

$$\int_{S+S'} \boldsymbol{A} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_V \nabla \cdot \boldsymbol{A} \, \mathrm{d}V \, . \tag{3.2.35}$$

Usually the divergence will be a bit easier to integrate.[†] Now we just have to find the second term of Eq. (3.2.34). But, we've made the problem a lot simpler, since on S', the unit normal is just $-\boldsymbol{e}_z$! This technique may not make things that much easier, but it's worth a shot if the open surface S has a complementary surface with, e.g., a convenient normal vector.

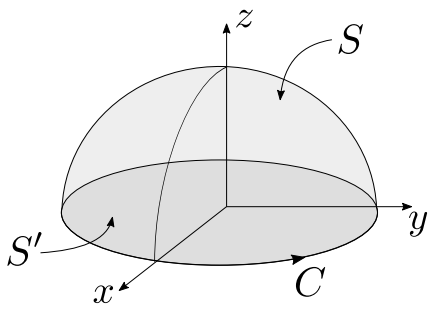


Figure 3.3: Integration geometry for evaluating Stokes' theorem using the divergence theorem. The surfaces S (bounded by C) and S' combine to form a closed hemisphere.

Areas and Volumes

Many times a problem invloving Stokes' or the divergence theorem will reduce to a problem of finding surface area or volume. To this end, memorizing the formulas in Table 3.1 may be useful.

3.3 Linear Algebra

3.3.1 Vector Spaces and Subspaces

We'll call an n-tuple of real numbers $\boldsymbol{v} \in \mathbb{R}^n$ a *vector*. A *vector space* is a nonempty set of objects which is closed under addition and scalar multiplication,[13]

[†]To wit, note that if we had found a vector potential for $\boldsymbol{A}$, the integrand would be 0, since $\nabla \cdot \nabla \times \boldsymbol{B} = 0$.

Shape	Area
Rectangle	$w \cdot h$
Trapezoid	$h \cdot \frac{1}{2}(w_1 + w_2)$
Triangle	$\frac{1}{2}b \cdot h$
Circle	πr^2
Ellipse	πab

Shape	Volume
Prism	$A \cdot h$
Pyramid	$\frac{1}{3}A \cdot h$
Sphere	$\frac{4}{3}\pi r^3$
Ellipsoid	$\frac{4}{3}\pi abc$

Table 3.1: Surface Areas/Volumes of Common Shapes

and has multiplicative and additive identities. That is to say, for vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ that are in the vector space V, and for any real scalars α and β, we have

1. Commutivity $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$
2. Associativity $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$
3. Multiplicative Distribution $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y}$
4. Multiplicative Association $(\alpha\beta)\boldsymbol{x} = \alpha(\beta\boldsymbol{x}) = \beta(\alpha\boldsymbol{x})$
5. Additive Identity $\boldsymbol{x} + \boldsymbol{0} = \boldsymbol{x}$
6. Multiplicative Identities $1\boldsymbol{x} = \boldsymbol{x}$ and $0\boldsymbol{x} = \boldsymbol{0}$

A subspace H of V is called a *subset* of V ($H \subseteq V$) it it itself is a vector space, and a *proper subset* ($H \subset V$) if H and V are not the same space.

We can define a dot product of vectors $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$ by

$$\boldsymbol{x} \cdot \boldsymbol{y} \equiv \sum_{i=1}^{n} x_n y_n = \|\boldsymbol{x}\|\|\boldsymbol{y}\| \cos\theta, \tag{3.3.1}$$

where θ is the angle between them. To show that the second expression is true, define $\boldsymbol{z} = \boldsymbol{x} - \boldsymbol{y}$, then from the law of cosines

$$\|\boldsymbol{z}\|^2 = \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\|\boldsymbol{x}\|\|\boldsymbol{y}\| \cos\theta, \tag{3.3.2}$$

But

$$\begin{aligned} \|\boldsymbol{z}\|^2 = \boldsymbol{z} \cdot \boldsymbol{z} &= (\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \\ &= \|\boldsymbol{x}\|^2 + \|\boldsymbol{y}\|^2 - 2\boldsymbol{x} \cdot \boldsymbol{y}\,. \end{aligned} \tag{3.3.3}$$

Comparing Eqs. (3.3.2) and (3.3.3) implies Eq. (3.3.1).

For vectors in $\mathbb{R}^3$, we can define a cross product: for $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^3$,

$$\boldsymbol{x} \times \boldsymbol{y} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - y_2 x_3), \boldsymbol{e}_x - (x_1 y_3 - y_1 x_3), \boldsymbol{e}_x + (x_1 y_2 - y_1 x_2), \boldsymbol{e}_z \,. \tag{3.3.4}$$

It turns out that

$$\|\boldsymbol{x} \times \boldsymbol{y}\| = \|\boldsymbol{x}\| \|\boldsymbol{y}\| \sin\theta \,. \tag{3.3.5}$$

This proof is a bit longer, but to show this, use Lagrange's identity

$$\|\boldsymbol{x} \times \boldsymbol{y}\|^2 = (\|\boldsymbol{x}\| \|\boldsymbol{y}\|)^2 - (\boldsymbol{x} \cdot \boldsymbol{y})^2 \,. \tag{3.3.6}$$

Then use Eq. (3.3.1) and $\cos^2\theta = 1 - \sin^2\theta$ to complete the proof. Note that the cross product is *not associative*, and that

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}) - \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) \,. \tag{3.3.7}$$

3.3.2 Linear Independence

A set of vectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_N\} \in \mathbb{R}^n$ is said to be *linearly independent* if there exists no non-trivial solution to

$$c_1 \boldsymbol{v}_1 + c_2 \boldsymbol{v}_2 + \cdots + c_N \boldsymbol{v}_N = \boldsymbol{0}. \tag{3.3.8}$$

In other words, the only set of coefficients that will make Eq. (3.3.8) true is $c_1 = c_2 = \cdots = c_N = 0$.

Test for Linear Independence

We can check if the set of vectors $\boldsymbol{v}_n$ is linearly independent by defining the vector $\boldsymbol{c} = (c_1, c_2, \ldots, c_N)$, and then the criterion for linear Independence can be written

$$\underbrace{\begin{pmatrix} \boldsymbol{v}_1^\mathsf{T} & \boldsymbol{v}_2^\mathsf{T} & \ldots & \boldsymbol{v}_N^\mathsf{T} \end{pmatrix}}_{M} \boldsymbol{c} = \boldsymbol{0} \,. \tag{3.3.9}$$

Assume $\det M$ is not 0. Then the only way to satisfy Eq. (3.3.9) would be to set $\boldsymbol{c} = 0$. In other words, there is no nontrivial set of c_n's to make Eq. (3.3.8) true. Thus if the determinant of M is non-zero, then its columns are linearly independent.

3.3.3 Orthogonality of Vectors and Subspaces

Two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are said to be *orthogonal* if their inner product vanishes; that is if

$$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0\,. \tag{3.3.10}$$

Two subspaces V_1 and V_2 are said to be orthogonal if every element in V_1 is orthogonal to every element in V_2; that is if

$$\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0 \quad \forall\, \boldsymbol{v}_1 \in V_1,\, \boldsymbol{v}_2 \in V_2\,. \tag{3.3.11}$$

3.3.4 Bases and Gram-Schmidt

The *span* of a set of n vectors is all linear combinations thereof:

$$\text{span}\,\{\boldsymbol{v}_i\} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \ldots + c_n\boldsymbol{v}_n\,. \tag{3.3.12}$$

If $\boldsymbol{v}_i$ are linearly independent, then they span $\mathbb{R}^n$. In this case we have a *basis*. Any vector $\boldsymbol{v} \in \mathbb{R}^n$ can be written as a sum of these basis vectors. Probably the most familiar basis is that of $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$ in $\mathbb{R}^3$.

These form an orthonormal basis, since $\boldsymbol{e}_j \cdot \boldsymbol{e}_k = \delta_{jk}$. Suppose we have a vector in a vector space $\boldsymbol{x} \in \mathbb{R}^m$, and we want to know the vector that's closest to it in another vector space $V = \mathbb{R}^n$ (with $n < m$). If we have an orthonormal basis of a V with unit vectors $\hat{\boldsymbol{v}}_i$, then the *projection* of $\boldsymbol{x}$ onto the subspace V is

$$\text{proj}_V\,\boldsymbol{x} = (\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_1)\,\hat{\boldsymbol{v}}_1 + (\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_2)\,\hat{\boldsymbol{v}}_2 + \ldots + (\boldsymbol{x} \cdot \hat{\boldsymbol{v}}_n)\,\hat{\boldsymbol{v}}_n\,. \tag{3.3.13}$$

Basis vectors don't have to have unit length or even be orthogonal; all they have to do is span the space. But as we saw with projection, having an orthonormal space makes thins a lot easier to work with. The *Gram-Schmidt process* is a way to find an orthonormal basis given n linearly independent vectors in $\mathbb{R}^n$. Since they are linearly independent, they span the space, are thus a basis. To find an orthonormal basis:

1. Normalize the first vector $\boldsymbol{n}_1 = \boldsymbol{v}_1/\|\boldsymbol{v}_1\|$
2. Create intermediate vector $\boldsymbol{y}_2$ from $\boldsymbol{v}_2$ by subtracting its component along $\boldsymbol{n}_1$

   $$\boldsymbol{y}_2 = \boldsymbol{v}_2 - (\boldsymbol{v}_2 \cdot \boldsymbol{n}_1)\,\boldsymbol{n}_1\,.$$

3. Normalize $\boldsymbol{n}_2 = \boldsymbol{y}_2/\|\boldsymbol{y}_2\|$.

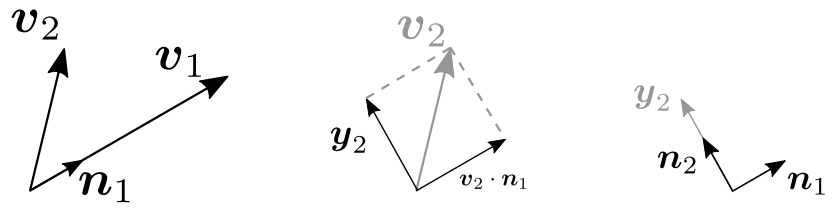


Figure 3.4: Illustration of the Gram-Schmidt procedure in $\mathbb{R}^2$.

4. Repeat, subtracting out all previous components; for example

$$\boldsymbol{y}_3 = \boldsymbol{v}_3 - (\boldsymbol{v}_3 \cdot \boldsymbol{n}_1)\, \boldsymbol{n}_1 - (\boldsymbol{v}_3 \cdot \boldsymbol{n}_2)\, \boldsymbol{n}_2 \,.$$

This process is extensible to arbitrary dimension, but the 2D example shown in Fig. 3.4 illustrates the technique.

3.3.5 Properties of the Determinant

The *determinant* of a square matrix is a bit unwieldy to write generally, so we'll just write it for small matrices. For a 2-by-2 matrix, the determinant is

$$\begin{aligned} \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ &= a_{11}a_{22} - a_{12}a_{21} \,. \end{aligned} \tag{3.3.14}$$

For a 3-by-3 matrix, the formula's a lot longer, but can be computed by taking the top row as coefficients, and multiplying them by the determinant of the matrix that remains when that element's row and column are excluded (and multiplying by ± 1, see below). That is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} . \tag{3.3.15}$$

In fact, this will work with any row as the weights, provided you use the correct sign for each coefficient. The signs are in a checkerboard pattern

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \tag{3.3.16}$$

Some useful properties of the determinant:

1. $\det \boldsymbol{1} = 1$
2. $\det A^{\mathsf{T}} = \det A$
3. $\det A^{-1} = \frac{1}{\det A}$
4. $\det AB = \det A \det B$ (A and B have same dimension)
5. $\det cA = c^n \det A$ (where $c = \text{const.}$ and $n = \dim A$)
6. $\det A = \prod a_{ii}$ (A is triangular)

3.3.6 Properties of Transposition

The *transpose* of an matrix is obtained by exchanging the rows and columns of a matrix. Thus if $A \in \mathbb{R}^{m\times n}$, then $A^{\mathsf{T}} \in \mathbb{R}^{n\times m}$ Some useful properties of the transposition are:

1. $(A + B)^{\mathsf{T}} = A^{\mathsf{T}} + B^{\mathsf{T}}$
2. $(\alpha A)^{\mathsf{T}} = \alpha A^{\mathsf{T}} + B^{\mathsf{T}}$
3. $(AB)^{\mathsf{T}} = B^{\mathsf{T}} A^{\mathsf{T}}$
4. $\operatorname{rank} A^{\mathsf{T}} = \operatorname{rank} A$

3.3.7 Eigenvalues and Eigenvectors

The eigenvectors of a square matrix A are the vectors $\boldsymbol{v}$ such that

$$A\boldsymbol{v} = \lambda \boldsymbol{v}, \tag{3.3.17}$$

where λ is the associated eigenvalue. That is, when a matrix operates on one of its eigenvectors, the result is the eigenvector $\boldsymbol{v}$ scaled by its eigenvalue λ.

Determining Eigenvalues

In order to determine the eigenvalues of a matrix A, begin with the definition Eq. (3.3.17). Then rearrange to write

$$(A - \lambda \boldsymbol{1})\,\boldsymbol{v} = \boldsymbol{0}, \tag{3.3.18}$$

where $\boldsymbol{1}$ is the identity matrix. Now since $\boldsymbol{x} \neq \boldsymbol{0}$, Eq. (3.3.18) indicates we want to find the value of λ such that $(A - \lambda \boldsymbol{1})$ is not invertible.[13] That is, we need to find the roots of

$$\boxed{\det (A - \lambda \boldsymbol{1}) = 0\,.} \tag{3.3.19}$$

Now if the characteristic equation is proportional to $(\lambda - a)^n$, then we say that a is an eigenvalue of A with **algebraic multiplicity** n.

Determining Eigenvectors

In order to determine the eigenvectors of a matrix A which correspond to a given eigenvalue λ we need to solve Eq. (3.3.18) (repeated here)

$$\boxed{(A - \lambda \mathbf{1})\boldsymbol{v} = \mathbf{0}} \tag{3.3.20}$$

for each eigenvalue λ. If an eigenvalue gives n linearly independent eigenvectors when Eq. (3.3.20) solved, then that eigenvector is said to have **geometric multiplicity** of n.

The following properties of eigenvalues and vectors may be useful:

1. The the trace of a matrix is equal to sum of the eigenvalues

$$\operatorname{tr} A = \sum \lambda \tag{3.3.21}$$

2. The determinant of a matrix is equal to the product of its eigenvalues

$$\det A = \prod \lambda \tag{3.3.22}$$

3. The square of a matrix has the same eigenvectors, and its eienvalues are the square of the original eigenvalues, since

$$\begin{aligned} A\boldsymbol{x} &= \lambda \boldsymbol{x} \\ AA\boldsymbol{x} &= \lambda (A\boldsymbol{x}) = \lambda (\lambda \boldsymbol{x}) \\ A^2\boldsymbol{x} &= \lambda^2 \boldsymbol{x}\,. \end{aligned} \tag{3.3.23}$$

3.3.8 Matrix Terminology

It will be probably be useful to recall the following definitions of matrix types, and the properties of each. All properties apply to real matrices (though many of these properties can be extended to the more general case, see the section on complex matrices).

Orthogonality

An **orthogonal matrix** is a *square* matrix A for which

$$A^{\mathsf{T}} = A^{-1}. \tag{3.3.24}$$

This implies that

$$A^{\mathsf{T}}A = AA^{\mathsf{T}} = \mathbf{1}. \tag{3.3.25}$$

The rows and columns of a symmetric matrix form an orthonomal basis. To see this, call the columns of A by $\boldsymbol{a}_n$. Then

$$A^{-1}A = A^{\mathsf{T}}A = \begin{pmatrix} — & \boldsymbol{a}_1^{\mathsf{T}} & — \\ — & \boldsymbol{a}_2^{\mathsf{T}} & — \\ & \vdots & \\ — & \boldsymbol{a}_N^{\mathsf{T}} & — \end{pmatrix} \begin{pmatrix} | & | & & | \\ \boldsymbol{a}_1 & \boldsymbol{a}_2 & \dots & \boldsymbol{a}_N \\ | & | & & | \end{pmatrix} = \mathbf{1}. \tag{3.3.26}$$

In other words,

$$\begin{pmatrix} \boldsymbol{a}_1^{\mathsf{T}} \cdot \boldsymbol{a}_1 & \boldsymbol{a}_1^{\mathsf{T}} \cdot \boldsymbol{a}_2 & \dots & \boldsymbol{a}_1^{\mathsf{T}} \cdot \boldsymbol{a}_N \\ \boldsymbol{a}_2^{\mathsf{T}} \cdot \boldsymbol{a}_1 & \boldsymbol{a}_2^{\mathsf{T}} \cdot \boldsymbol{a}_2 & & \vdots \\ \vdots & & \ddots & \\ \boldsymbol{a}_N^{\mathsf{T}} \cdot \boldsymbol{a}_1 & \dots & & \boldsymbol{a}_N^{\mathsf{T}} \cdot \boldsymbol{a}_N \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & & 1 \end{pmatrix}. \tag{3.3.27}$$

But this means that $\boldsymbol{a}_m^{\mathsf{T}} \cdot \boldsymbol{a}_n = \delta_{mn}$, where δ_{mn} is the Kronecker delta. This the definition of an orthonormal set. Note that if we view an orthogonal matrix as a transformation, that the length and angles between vectors are preserved. To see this, note that if A is orthogonal,

$$\|A\boldsymbol{x}\|^2 = (A\boldsymbol{x})^{\mathsf{T}}(A\boldsymbol{x}) = \boldsymbol{x}^{\mathsf{T}}A^{\mathsf{T}}A\boldsymbol{x} = \boldsymbol{x}^{\mathsf{T}}\underbrace{A^{-1}A}_{\mathbf{1}}\boldsymbol{x} = \boldsymbol{x}^{\mathsf{T}}\boldsymbol{x} = \|\boldsymbol{x}\|^2. \tag{3.3.28}$$

Similarly, the angle θ between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ can be found from their dot product

$$\cos\theta = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|}. \tag{3.3.29}$$

The angle ϕ between the transformed vectors is then

$$\begin{aligned}\cos\phi &= \frac{(A\boldsymbol{x})\cdot(A\boldsymbol{y})}{\|(A\boldsymbol{x})\|\|(A\boldsymbol{y})\|} \\ &= \frac{(A\boldsymbol{x})^{\mathsf{T}}(A\boldsymbol{y})}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|} \qquad \text{[Eq. (3.3.28)]} \\ &= \frac{\boldsymbol{x}^{\mathsf{T}}\left(A^{\mathsf{T}}A\right)\boldsymbol{y}}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|} \\ &= \frac{\boldsymbol{x}^{\mathsf{T}}\boldsymbol{y}}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|} = \frac{\boldsymbol{x}\cdot\boldsymbol{y}}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|} = \cos\theta\,. \end{aligned} \tag{3.3.30}$$

For an $n\times n$ orthogonal matrix A:

- The columns of A form an orthonormal basis for $\mathbb{R}^n$
- The eigenvalues of A have magnitude 1 ($\|\lambda\| = 1$)
- $\det A = \pm 1$

Symmetry

A *square* matrix A is said to be **symmetric** if

$$A^{\mathsf{T}} = A\,. \tag{3.3.31}$$

For an $n\times n$ symmetric matrix A, the following are necessary (not necessarity sufficient) conditions:

- A has real eigenvalues
- Eigenvectors of A corresponding to different eigenvalues are orthogonal
- A has n linearly independent eigenvectors
- $\det A = \det A^{\mathsf{T}}$
- $\det A = \det -A$
- $\det A = (-1)^n \det A$

These are consequences of the properties of the determinant (see Sec. 3.3.5). To see that the eigenvalues must be real,[†] consider the eigenvalue λ with

[†]Since complex vectors are generally not considered, the specifics of this proof are likely out of scope. However, the techniques apply to other questions from past exams. Presentation here follows notes on "Basic Matrix Theorems" by Richard Quandt.

corresponding eigenvector $\boldsymbol{v}$. Then,

$$\begin{aligned} A\boldsymbol{v} &= \lambda \boldsymbol{v} \\ A^*\boldsymbol{v}^* = A\boldsymbol{v}^* &= \lambda^*\boldsymbol{v}^* \quad (A \text{ is real}) \\ \boldsymbol{v}^{\mathsf{T}} A\boldsymbol{v}^* &= \lambda^* \boldsymbol{v}^{\mathsf{T}}\boldsymbol{v}^* \,. \end{aligned} \tag{3.3.32}$$

Then since

$$\boldsymbol{v}^\dagger A\boldsymbol{v} = \lambda \boldsymbol{v}^\dagger \boldsymbol{v}\,, \tag{3.3.33}$$

subtracting Eq. (3.3.32) from Eq. (3.3.33) gives

$$\boldsymbol{v}^\dagger A\boldsymbol{v} - \boldsymbol{v}^{\mathsf{T}} A\boldsymbol{v}^* = \lambda \boldsymbol{v}^\dagger \boldsymbol{v} - \lambda^* \boldsymbol{v}^{\mathsf{T}}\boldsymbol{v}^* \,. \tag{3.3.34}$$

But consider the left side of Eq. (3.3.34). Since A is symmetric and real, it doesn't matter whether we take the conjugate of $\boldsymbol{v}$ for left or right multiplying. That is, for any symmetric matrix $A \in \mathbb{R}^{n\times n}$ and vector $\boldsymbol{a} \in \mathbb{C}^n$,

$$\boldsymbol{a}^{\mathsf{T}} A\boldsymbol{a}^* = \boldsymbol{a}^\dagger A\boldsymbol{a}\,. \tag{3.3.35}$$

Thus, the left side of Eq. (3.3.34) vanishes. Therefore

$$(\lambda - \lambda^*)\, \boldsymbol{v}^\dagger \boldsymbol{v} = 0\,. \tag{3.3.36}$$

Since $\boldsymbol{v} \neq \boldsymbol{0}$ is an eigenvector of A, $\boldsymbol{v}^\dagger \boldsymbol{v} > 0$, and thus

$$\lambda = \lambda^* \quad \rightarrow \quad \lambda \in \mathbb{R}\,. \tag{3.3.37}$$

To see that the eigenvectors must be othogonal, let $\boldsymbol{v}_1$ and $\boldsymbol{v}_1$ be eigenvectors of A corrseponding to different eigenvalues λ_1 and λ_2. Then consider

$$\lambda_1 \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = (A\boldsymbol{v}_1)^{\mathsf{T}} \cdot \boldsymbol{v}_2 = \boldsymbol{v}_1 \cdot \left(A^{\mathsf{T}} \boldsymbol{v}_2\right) = \boldsymbol{v}_1 \cdot (A\boldsymbol{v}_2)\,, \tag{3.3.38}$$

since $A^{\mathsf{T}} = A$. Then, since $A\boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_2$,

$$\lambda_1 \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = \boldsymbol{v}_1 \cdot (\lambda_2 \boldsymbol{v}_2) = \lambda_2 \boldsymbol{v}_1 \cdot \boldsymbol{v}_2\,. \tag{3.3.39}$$

Equation (3.3.39) implies

$$(\lambda_1 - \lambda_2)\, \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0\,. \tag{3.3.40}$$

Since we've said that $\lambda_1 \neq \lambda_2$, we conclude that $\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 = 0$, i.e., that the eigenvectors are orthogonal.

Skew Symmetry

A *square* matrix A is said to be **skew symmetric** (or **antisymmetric**) if

$$A^{\mathsf{T}} = -A\,. \tag{3.3.41}$$

Note that this means the main diagonal must be zero (since $A_{ii} = -A_{ii}$). For an $n\times n$ skew-symmetric matrix A, the following are necessary (but not necessarily sufficient):

- The eigenvalues of A are 0 or purely imaginary
- $\det A = \det A^{\mathsf{T}}$
- $\det A = \det -A$
- $\det A = (-1)^n \det A$

These last of these are consequences of the properties of the determinant (see Sec. 3.3.5).

Similar to the symmetric case, we can show that the eigenvalues must be 0 or purely imaginary. Suppose as before that $\boldsymbol{v} \in \mathbb{C}^n$ is an eigenvector of the skew symmetric matrix A with eigenvalue λ. Then

$$(A\boldsymbol{v}) \cdot \boldsymbol{v} = (\lambda \boldsymbol{v}) \cdot \boldsymbol{v} = \lambda^* \boldsymbol{v}^\dagger \boldsymbol{v} = \lambda \|\boldsymbol{v}\|^2\,. \tag{3.3.42}$$

By the same procedure

$$-\boldsymbol{v} \cdot (A\boldsymbol{v}) = -\boldsymbol{v} \cdot (\lambda \boldsymbol{v}) = -\lambda^* \boldsymbol{v} \cdot \boldsymbol{v} = -\lambda^* \|\boldsymbol{v}\|^2\,. \tag{3.3.43}$$

Equations (3.3.42) and (3.3.43) imply that

$$(\lambda + \lambda^*)\,\|\boldsymbol{v}\|^2 = 0\,. \tag{3.3.44}$$

Since $\boldsymbol{v}$ is an eigenvector, it cannot have magnitude 0, and so $\lambda = -\lambda^*$, which implies the real part of λ must vanish.

Note that any square matrix A can be written as the sum of a symmetric and antisymmetrix matrix:

$$A = \frac{1}{2}\underbrace{\left(A - A^{\mathsf{T}}\right)}_{\text{antisymmetric}} + \frac{1}{2}\underbrace{\left(A + A^{\mathsf{T}}\right)}_{\text{symmetric}}\,. \tag{3.3.45}$$

Similarity

Two square matrices A and B are said to be *similar* if there exists some matrix S such that

$$B = S^{-1}AS \qquad \Longrightarrow \qquad SB = AS\,. \tag{3.3.46}$$

It turns our that all matrices $A \in \mathbb{R}^{n\times n}$ with n linearly independent eigenvalues is similar to a diagonal matrix (see section on **Diagonalization**).

Positive Definiteness

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be **positive definite** if

$$\boldsymbol{a}^{\mathsf{T}} A\boldsymbol{a} > 0 \tag{3.3.47}$$

for all non-zero $\boldsymbol{a} \in \mathbb{R}^n$. A positive-definite matrix A has the following necessary *and sufficient* properties:

- Its eigenvalues are all positive.
- It may be written $A = R^{\mathsf{T}}R$, where R has independent columns.

Similarly, if $\boldsymbol{a}^{\mathsf{T}} A\boldsymbol{a} < 0$, then A is **negative definite** and has negative eigenvalues. If the strict inequalities become $\geq$ and $\leq$, then the matrices are called **positive semidefinite** and **negative semidefinite**, and their eigenvalues are nonnegative and nonpositive, respectively.

These properties are not terribly difficult to show. For the first, suppose $\boldsymbol{v}$ is an eigenvector of the positive definite matrix A with eigenvalue λ. Then

$$\begin{aligned} \boldsymbol{v}^{\mathsf{T}}(A\boldsymbol{v}) &= \boldsymbol{v}^{\mathsf{T}}(\lambda\boldsymbol{v}) \\ \boldsymbol{v}^{\mathsf{T}}A\boldsymbol{v} &= \lambda\boldsymbol{v}^{\mathsf{T}}\boldsymbol{v} = \lambda\|\boldsymbol{v}\|^2\,. \end{aligned} \tag{3.3.48}$$

Note the left side of Eq. (3.3.48) is strictly positive by definition, and $\|\boldsymbol{v}\|^2 > 0$, since $\boldsymbol{v} \neq \boldsymbol{0}$ is an eigenvector. We conclude that $\lambda > 0$.

For the second, note that if we write $A = R^{\mathsf{T}}R$, where the columns of R are linearlt independent. Note that we haven't yet assumed A is positive definite. Then

$$\boldsymbol{v}^{\mathsf{T}}A\boldsymbol{v} = \boldsymbol{v}^{\mathsf{T}}\left(R^{\mathsf{T}}R\right)\boldsymbol{v} = \left(\boldsymbol{v}^{\mathsf{T}}R^{\mathsf{T}}\right)(R\boldsymbol{v}) = (R\boldsymbol{v})\,(R\boldsymbol{v}) = \|R\boldsymbol{v}\|^2\,. \tag{3.3.49}$$

If the columns of R are linearly independent, the $R\boldsymbol{v} = \boldsymbol{0}$ if and only if $\boldsymbol{0}$. Thus $\|R\boldsymbol{v}\|^2$ is strictly positive for nonzero $\boldsymbol{v}$ and so too must be $\boldsymbol{v}^\mathsf{T} A\boldsymbol{v}$. Thus A is positive definite.[†]

Diagonalization

It is often useful to be able to write a matrix A in the form

$$A = PDP^{-1}, \tag{3.3.50}$$

where D is a diagonal matrix. Why? Well, suppose we want to compute A^n, where n is, say, 121. Performing the computation would take way longer than the allotted time on the exam. But, if we write A in the form of Eq. (3.3.50), then we have

$$\begin{aligned} A^n &= \underbrace{\left(PDP^{-1}\right)\left(PDP^{-1}\right)\ldots\left(PDP^{-1}\right)}_{n \text{ times}} \\ &= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)\ldots(P^{-1}P)DP^{-1} \\ &= PD^nP^{-1}. \end{aligned} \tag{3.3.51}$$

And D^n is easy to compute: just raise each diagonal element to the n^{th} power.

Or suppose we have a transformation

$$\boldsymbol{y} = A\boldsymbol{x}. \tag{3.3.52}$$

In general, it's difficult to say what A will do to $\boldsymbol{x}$. But it we diagonalize A, we can write

$$\boldsymbol{y} = \left(PDP^{-1}\right)\boldsymbol{x}, \tag{3.3.53}$$

and then define $\boldsymbol{x}' = P^{-1}\boldsymbol{x}$ and $\boldsymbol{y}' = P^{-1}\boldsymbol{y}$, so that

$$\begin{aligned} \boldsymbol{y} &= PDP^{-1}\boldsymbol{x} \\ P^{-1}\boldsymbol{y} &= P^{-1}PD\boldsymbol{x}' \\ \boldsymbol{y}' &= D\boldsymbol{x}'. \end{aligned} \tag{3.3.54}$$

[†]Second proof follows notes by Dylan Zwick (who is an expert on tropical matrices, whatever the heck those are).

Equation (3.3.54) is clearly just a scaling. Note too that if A is symmetric, then $P^{-1} = P^{\mathsf{T}}$, and we can transform $\boldsymbol{x}$ easily.[†]

How do we find P and D though? It turns out that these are just the eigenvectors

$$P = \begin{pmatrix} | & | & & | \\ \boldsymbol{x}_1 & \boldsymbol{x}_2 & \dots & \boldsymbol{x}_n \\ | & | & & | \end{pmatrix}, \tag{3.3.55}$$

and eigenvalues

$$D = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}. \tag{3.3.56}$$

Diagonalizability

Not all matrices can be written in the form of Eq. (3.3.50). The following are true

- A matrix is diagonalizable if and only if it has n linearly independent eigenvectors.
- If a matrix has n distinct eigenvalues, then it is diagonalizable. **NOTE:** A diagnoalizable matrix *does not necessarily* have n distinct values.
- Symmetric matrices are always diagonalizable, and $P^{-1} = P^{\mathsf{T}}$ (this follows from the properties of symmetric matrices, see **Symmetry**.)

While Eq. (3.3.51) is probably the most common way to raise a matrix to a power, note that we have to find the eigenvectors to form P. We can also invoke

[†]Symmetric matrices in $\mathbb{R}^2$ correspond to quadratic forms; notice in this case

$$\boldsymbol{x}^{\mathsf{T}} A \boldsymbol{x} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2,$$

which can be an ellipse, parabola, hyperbola, or circle (or a line if $b = 0$).

the Cayley–Hamilton theorem, which states that a matrix obeys its own characteristic polynomial. That is, if for some matrix A, we have a characteristic polynomial

$$\det(A - \lambda \mathbf{1}) = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0 = 0, \tag{3.3.57}$$

then it is true that

$$A^n + c_{n-1}A^{n-1} + \ldots + c_1 A + c_0 \mathbf{1} = 0. \tag{3.3.58}$$

Now, through recurrence relationships, it can be shown that

$$A^k = b_1 A + b_0 \mathbf{1} \tag{3.3.59}$$

$$\implies \lambda_i^k = b_1\lambda_i + b_0. \tag{3.3.60}$$

Equation (3.3.60) will give a matrix equation that may be solved for b_0 and b_1, allowing us to find A^k from Eq. (3.3.59).

Matrix Exponentiation

We define the exponentiation of a matrix A as

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \mathbf{1} + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \ldots. \tag{3.3.61}$$

Note that if A is diagonalizable, we have

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \frac{\left(PDP^{-1}\right)^k}{k!} \\ &= P\left(\sum_{n=0}^{\infty} \frac{D^k}{k!}\right)P^{-1} \\ &= P\begin{pmatrix} e^{\lambda_1} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2} & & \vdots \\ \vdots & & \ddots & \\ 0 & \ldots & & e^{\lambda_n} \end{pmatrix}P^{-1}. \end{aligned} \tag{3.3.62}$$

Since one test asked for two methods of matrix exponentiation, we can also use some knowledge from systems of ODEs (see Sec. 3.4.4) and Laplace transforms (Sec. 3.4.7). The first step is to recognize that e^{At} is a solution to

$$\boldsymbol{x}' = A\boldsymbol{x}, \tag{3.3.63}$$

with initial condition $\boldsymbol{x}_0 = \boldsymbol{1}$. Then if we take the Laplace transform to get rid of the derivative, we have

$$\begin{aligned} \mathcal{L}\left[\boldsymbol{x}'(t)\right] &= \mathcal{L}\left[A\boldsymbol{x}(t)\right] \\ s\hat{\boldsymbol{x}}(s) - \boldsymbol{x}_0 &= A\hat{\boldsymbol{x}}(s) \\ (s\boldsymbol{1} - A)\,\hat{\boldsymbol{x}}(s) &= \boldsymbol{x}_0 = \boldsymbol{1} \\ \hat{\boldsymbol{x}}(s) &= (s\boldsymbol{1} - A)^{-1}\ . \end{aligned} \tag{3.3.64}$$

But we know the solution $\hat{\boldsymbol{x}}(s)$ is just the Laplace transform of e^{At}, so we just need to take the inverse transform

$$e^{At} = \mathcal{L}^{-1}\left[(s\boldsymbol{1} - A)^{-1}\right]\ . \tag{3.3.65}$$

Triangularization

A matrix is **triangualar** if all of its nonzero entries lie above (*upper triangular*) or below (*lower triangular*) its main diagonal (or on it). These matrices have the nice property that a triangular matrix times another triangular matrix is again triangular (e.g., $U_1U_2 = U_3$ and $L_1L_2 = L_3$).

Perhaps the most common application is to find a factorization that will allow us to solve an easier problem. For example, suppose we want to solve

$$A\boldsymbol{x} = \boldsymbol{b}\,. \tag{3.3.66}$$

If we can factor A as $A = LU$, then we can set

$$\boldsymbol{y} = U\boldsymbol{x} \tag{3.3.67}$$

and solve

$$L\boldsymbol{y} = \boldsymbol{b}\,. \tag{3.3.68}$$

Finding $\boldsymbol{y}$ is easy, since we can just forward substitute. Consider

$$L\boldsymbol{y} = \boldsymbol{b} \quad \rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ l_{21} & 1 & 0 & b_2 \\ l_{31} & l_{32} & 1 & b_3 \end{array}\right) \tag{3.3.69}$$

Once we know $\boldsymbol{y}$, we can find $\boldsymbol{x}$ with simple back substitution

$$U\boldsymbol{x} = \boldsymbol{y} \quad \rightarrow \quad \left(\begin{array}{ccc|c} u_{11} & u_{12} & u_{13} & y_1 \\ 0 & u_{22} & u_{23} & y_2 \\ 0 & 0 & u_{33} & y_3 \end{array}\right) \tag{3.3.70}$$

To find the LU factorization of a matrix, we simply set up the equation

$$A = LU$$
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \tag{3.3.71}$$

This looks kind of gnarly, but it turns out that solving for the elements of each matrix isn't too bad, since we're left with a forward substitution type problem.

For a triagnular (lower or upper) matrix $A \in \mathbb{R}^{n\times n}$:

- The eigenvalues of A are the entries of the main diagonal
- The product of its main diagonal entries is $\det A$

Complex Matrices

Many of the properties that apply to real vectors and matrices apply when extended to complex numbers, provided we modify our use of the transpose and inner product. For complex matrices, the transpose operation must be accompanied by complex conjugation. This is sometimes called the adjoint

$$A^{\mathsf{T}} \to (A^*)^{\mathsf{T}} = \left(A^{\mathsf{T}}\right)^* = A^\dagger \,. \tag{3.3.72}$$

Similarly, the inner product has the property

$$\boldsymbol{u} \cdot \boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{u})^* \,. \tag{3.3.73}$$

3.4 Linear Ordinary Differential Equations

This section deals with linear ordinary differential equations (ODEs). A general linear ODE for the function $x(t)$ may be written as

$$D^n x + P(t)\, D^{n-1} x + \ldots + Q(t) Dx + R(t) = 0\,, \tag{3.4.1}$$

where the notation $D \equiv \mathrm{d}/\mathrm{d}t$ has been used. Linearity can be verified from Eq. (3.4.1), since if $x_1(t)$ and $x_2(t)$ are solutions, so to is $a\, x_1 + b\, x_2$, where a and b are constants.

3.4.1 Initial-Value Problems

An initial value problem is one that can be written

$$\frac{\mathrm{d}^n x}{\mathrm{d}t^n} = f\left(t, x, x', \ldots, x^{(n)}\right), \tag{3.4.2}$$

to be solved with n initial conditions

$$x(0) = x_0, \quad x'(0) = x_0', \quad \ldots, \quad x^{(n)}(0) = x_0^{(n)}. \tag{3.4.3}$$

Of the most interest here is lower-order equations, i.e., $n \sim 3$.

3.4.2 Two-point Boundary-Value Problems

Boundary value problems comprise the set of problems where the ODE is to be solved subject to boundary conditions.

3.4.3 Homogeneous and Nonhomogeneous Solutions

If in Eq. (3.4.1), $R(t)$ is 0, then the equation is homogeneous. If $R(t)$ does not vanish in Eq. (3.4.1), then the equation is nonhomogeneous. The method for finding the solution to a nonhomogeneous is to first solve the homogeneous problem. That is, suppose our ODE is

$$L[x(t)] = f(t), \tag{3.4.4}$$

where $f(t) = -R(t)$ and L is some linear operator, such that we have an equation of the form of Eq. (3.4.1). We first solve the associated homogeneous problem

$$L[x_h(t)] = 0, \tag{3.4.5}$$

which is generally a bit easier. Next we seek a particular solution $x_p(t)$ which is a solution to the equation

$$L[x_p(t)] = f(t), \tag{3.4.6}$$

Then, since our ODE is linear, we can write

$$L[x_h(t) + x_p(t)] = L[x_h(t)] + L[x_p(t)] = 0 + f(t) = f(t). \tag{3.4.7}$$

Thus $x(t) = x_h(t) + x_p(t)$ is the full solution to the problem.

The best approach for finding the solution depends on the precise form of the coefficients, the nonhomogeneous part $R(t)$, and the order of the equation. Some techniques are described in Secs. 3.4.4 and 3.4.6.

3.4.4 First Order Solution Techniques

If the ODE of interest contains only a first derivative, we have a few lines of attack. They are presented in approximate order of complexity, though there may be more than one way of approaching it. A summary of the forms and their most readilly applicable solution method is shown in Fig. 3.5.

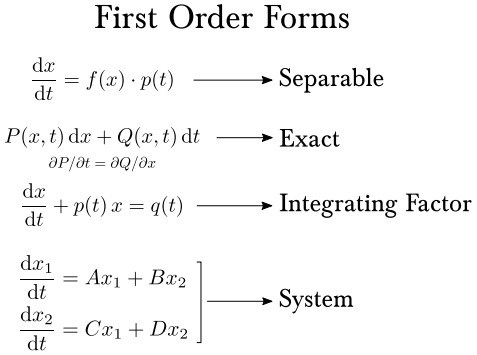


Figure 3.5: First order ODE forms and their most natural solution techniques.

Separable Equations

Perhaps the most easily solved are equations of the form

$$\frac{dx}{dt} = f(x) \cdot p(t)\,. \tag{3.4.8}$$

In this case we can just separate variables and write

$$\frac{dx}{f(x)} = p(t)\,dt \tag{3.4.9}$$

and integrate directly. The constant of integration we get in Eq. (3.4.9) will be determined by our initial condition.

Exact ODEs

Suppose there exists some continuously differentiable potential function $F(x,t)$,[†] whose exact differential 0

$$dF = \frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial t}\,dt = 0\,. \tag{3.4.10}$$

[†]Precisely, F must be continuously differentiable only on some simply connected, open subset of $\mathbb{R}^2$, so we could handle discontinuities by solving in individual regions. But anyway.

Since $x = x(t)$, Eq. (3.4.10) is a differential equation and is called **exact**. Usually, we want to find this potential function, so that we can say a solution is

$$F = C\,, \tag{3.4.11}$$

so we will first need to determine if such a function even exists. Since we've required F to be continuously differentiable, it and all its derivatives must be continuous. Then we can invoke Schwartz' theorem, which says that

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}\,. \tag{3.4.12}$$

That is, we see that some differential equation

$$P(x,t)\,\mathrm{d}x + Q(x,t)\,\mathrm{d}t = 0 \tag{3.4.13}$$

is exact if and only if

$$\frac{\partial P}{\partial t} = \frac{\partial Q}{\partial x}\,. \tag{3.4.14}$$

To find F, notice that

$$\begin{aligned} \mathrm{d}F &= \frac{\partial F}{\partial x}\,\mathrm{d}x + \frac{\partial F}{\partial y}\,\mathrm{d}y \\ &= 0 \quad [\text{from Eq. (3.4.10)}] \\ &\implies F = C\,. \end{aligned} \tag{3.4.15}$$

Then we can integrate P and Q:

$$P(x,y) = \frac{\partial F}{\partial x} \implies F = \int P\,\mathrm{d}x + f(y), \tag{3.4.16}$$

and

$$Q(x,y) = \frac{\partial F}{\partial y} \implies F = \int Q\,\mathrm{d}y + g(x), \tag{3.4.17}$$

and the functions $g(x)$ and $f(y)$ are determined by comparison of Eqs. (3.4.16) and (3.4.17)

Integrating Factors

Most generally, linear first-order ODEs have the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} + p(t)x = q(t)\,. \tag{3.4.18}$$

Suppose we define an **integrating factor**

$$\mu \equiv C\, e^{\int p(t)\,\mathrm{d}t}\,. \tag{3.4.19}$$

Now multiply Eq. (3.4.18) this integrating factor

$$\mu\frac{\mathrm{d}x}{\mathrm{d}t} + \mu\, p(t)x = \mu\, q(t)\,. \tag{3.4.20}$$

But, from Eq. (3.4.19)

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = C\, e^{\int p(t)\,\mathrm{d}t}\cdot\frac{\partial}{\partial t}\left(\int p(t)\,\mathrm{d}t\right) = p(t)\,\mu\,, \tag{3.4.21}$$

so that Eq. (3.4.20) is the same as

$$\begin{aligned}\mu\frac{\mathrm{d}x}{\mathrm{d}t} + \left(\frac{\mathrm{d}\mu}{\mathrm{d}t}\right)x &= \mu\, q(t)\\ \frac{\mathrm{d}}{\mathrm{d}t}(\mu\, x) &= \mu\, q(t)\\ \implies x(t) &= \frac{1}{\mu}\int \mu\, q(t)\,\mathrm{d}t\,.\end{aligned} \tag{3.4.22}$$

Linear Systems

Suppose we have a linear system of ODEs

$$\begin{aligned}\frac{\mathrm{d}x_1}{\mathrm{d}t} &= a_1x_1 + b_1x_2\\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= a_2x_1 + b_2x_2\,.\end{aligned} \tag{3.4.23}$$

We can write Eq. (3.4.23) as a matrix equation

$$D\boldsymbol{x} = A\boldsymbol{x}\,, \tag{3.4.24}$$

where $\boldsymbol{x} = (x_1,\, x_2)^{\mathsf{T}}$, $D = (\mathrm{d}/\mathrm{d}t,\, \mathrm{d}/\mathrm{d}t)$, and

$$A = \begin{pmatrix} a_1 & b_1\\ a_2 & b_2\end{pmatrix}\,. \tag{3.4.25}$$

The solution to Eq. (3.4.24) can be found from the eigenvalues and eigenvectors of the matrix A. If A has n linearly independent eigenvectors $\boldsymbol{v}_i$, which correspond to eigenvalues λ_i, then the solution to Eq. (3.4.24) is given by

$$\boldsymbol{x} = c_1 e^{\lambda_1 t}\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\boldsymbol{v}_2\,. \tag{3.4.26}$$

But what if we don't have a homogeneous system? That is, what if Eq. (3.4.24) is of the form

$$D\boldsymbol{x} = A\boldsymbol{x} + F\,, \tag{3.4.27}$$

where

$$F = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}? \tag{3.4.28}$$

Well in this case, we proceed use a variation of parameters technique (see Sec. 3.4.6) whereby we assume that the solution $\boldsymbol{x}$ is a sum of the homogeneous solution $\boldsymbol{x}_h$ and a particular solution $\boldsymbol{x}_p$ with the form

$$\begin{aligned} \boldsymbol{x}_p &= c_1(t)\boldsymbol{h}_1 + c_2\boldsymbol{h}_2 \\ &= c_1(t)\left(c_1 e^{\lambda_1 t}\boldsymbol{v}_1\right) + c_1(t)\left(c_2 e^{\lambda_2 t}\boldsymbol{v}_2\right)\,. \end{aligned} \tag{3.4.29}$$

3.4.5 Higher Order ODEs—Homogeneous Solutions

Higher order ODEs may be solved with the same process as first-order systems:

1. Find the homogeneous solution $L(x_h) = 0$
2. Find a the particular solution $L(x_p) = f(t)$

The general solution will be a sum of these homogeneous and particular solutions

$$x(t) = x_p(t) + x_h(t)\,. \tag{3.4.30}$$

Characteristic Equation

If the ODE has constant coefficients, the first line of attack for solving an ODE is usually to find its characteristic equation. Suppose we have a homogeneous ODE that may be written in the form of Eq. (3.4.1). If we can factor the operator into a form

$$(D - \lambda_1)(D - \lambda_2)\ldots(D - \lambda_n)x = 0\,, \tag{3.4.31}$$

then we can immediately write the solution as

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \ldots + C_n e^{\lambda_n t} \,. \tag{3.4.32}$$

Repeated Roots What if a root λ of the characteristic equation Eq. (3.4.32) has multiplicity n? In this case, the solutions can be found by multiplying the exponential solution by successive powers of t:

$$e^{\lambda},\ te^{\lambda t},\ \ldots,\ t^{n-1}e^{\lambda t} \,. \tag{3.4.33}$$

Cauchy-Euler

Suppose we have an ODE of the form

$$a_n t^n \frac{\mathrm{d}^n x}{\mathrm{d}t^n} + a_{n-1} t^{n-1} \frac{\mathrm{d}^{n-1} x}{\mathrm{d}t^{n-1}} + \cdots a_1 t \frac{\mathrm{d}x}{\mathrm{d}t} + a_0 x = 0 \,. \tag{3.4.34}$$

We'll assume a solution that looks like

$$x(t) = t^m. \tag{3.4.35}$$

We note then that

$$\begin{aligned} \mathrm{d}x/\mathrm{d}t &= m\, t^{m-1} \\ \mathrm{d}^2x/\mathrm{d}t^2 &= m(m-1)\, t^{m-2} \\ &\text{etc.} \end{aligned} \tag{3.4.36}$$

From Eq. (3.4.34), we see the terms that are powers of t will cancel, and we'll be left with an equation for m, called the **auxiliary equation**. For example, if the ODE is of degree 2, we'll recover

$$a_2\, m(m-1) + a_1\, m + a_0 = 0. \tag{3.4.37}$$

The n roots of Eq. (3.4.37) will give the general solution to Eq. (3.4.34).

Repeated Roots If a root λ of the auxiliary equation [Eq. (3.4.37)] appears more than once (suppose multiplicity 3), the solutions are given by

$$t^{\lambda}\,,\ t^{\lambda} \ln t,\ t^{\lambda} \ln^2 t \,. \tag{3.4.38}$$

Power Series Solutions

Any continuously differential function can be expressed as its Taylor series. It stands to reason then that we could find the solution to a differential equation by seeking a power series solution of the form

$$x(t) = \sum_{n=0}^{\infty} c_n t^n \,. \tag{3.4.39}$$

We'll consider only homogeneous[†] 2nd degree ODEs of the form

$$p(t)\, x'' + q(t)\, x' + r(t)\, x = 0 \,. \tag{3.4.40}$$

Note that a power series solution is guaranteed only on intervals that do not contain a root of $p(t)$. For solutions on intervals that contain such a root, see the next section on Frobenius Method.

We then substitute Eq. (3.4.39) into Eq. (3.4.40), which will typically yield a complicated algebra problem. Our end goal is to get a recurrence relationship, something like[‡]

$$c_{n+2} = f(n) c_n \,. \tag{3.4.41}$$

This will allow us to find power series proportional to two constants, usually c_0 and c_1. These are the two linearly independent solutions to the ODE.

Frobenius Method

Suppose we have an ODE of the form

$$p(t)\, x'' + q(t)\, x' + r(t)\, x = 0 \,, \tag{3.4.42}$$

where $p(t)$, $q(t)$, and $r(t)$ are polynomials in t. Our power series solution method from the previous section will fail if $p(t)$ is 0 anywhere in the region of interest. What if we want a solution in a region that includes that root of $p(t)$? We'll first rewrite Eq. (3.4.42) in *standard form*

$$x'' + Q(t) x' + R(t) x = 0 \,, \tag{3.4.43}$$

[†] A question involving a forcing function $f(t)$ on the right hand side of Eq. (3.4.40) is conceivable. In that case, it would almost certainly be a polynomial. If an $f(t)$ that is *not* a polynomial is given do two things: (1) expand $f(t)$ as a Taylor series and (2) email me and I'll mail you a dollar.

[‡] We could also find $c_{n+3} = f(n) c_n$ with $c_2 = 0$. The main thing is that for a second-order problem, we'll be left with two unknowns, giving two independent series solutions.

where

$$Q(t) = \frac{q(t)}{p(t)}, \quad \text{and} \quad R(t) = \frac{r(t)}{p(t)} \tag{3.4.44}$$

Clearly the root of $p(t)$, say $t = t_0$, is a *singular* point, since $P(t)$ and $Q(t)$ blow up at this point. In order for Frobenius' method to work, we need to ensure that $t = t_0$ is what's called a *regular singular point*. That is, we need to check that

$$(t - t_0)Q(t) \quad \text{and} \quad (t - t_0)^2 R(t) \tag{3.4.45}$$

remain finite at $t = t_0$, even though $Q(t)$ and $R(t)$ don't. More precisely, the requirement is that $Q(t)$ and $R(t)$ have convergent power series expansions about the singular point.

If both of Eqs. (3.4.45) do remain bounded, then we can use the method of Frobenius and assume a solution of the form

$$x(t) = t^r \sum_{n=0}^{\infty} c_n t^n \,. \tag{3.4.46}$$

Substitution of Eq. (3.4.46) into the governing ODE will allow us to solve for the b_ns.

The Frobenius method only guarantees one series solution, but often two may be found (in a manner very similar to the recurrence relations found using the **Power Series Solution**). Further, this series solution will converge to $x(t)$ within some finite radius of convergence R. But are two solutions guarenteed for this method as they are for a power series about an ordinary point?

It can be shown[†] that

$$r(r + 1) + a_0 r + b_0 = 0\,, \tag{3.4.47}$$

where a_0 and b_0 are the first terms of the series expansions

$$(t - t_0)Q(t) = a_0 + a_1 t + a_2 t^2 + \ldots \tag{3.4.48}$$

$$(t - t_0)^2 R(t) = b_0 + b_1 t + b_2 t^2 + \ldots \,. \tag{3.4.49}$$

[†]To show this, multiply Eq. (3.4.43) by $(t-t_0)^2$, and then substitute the result into the assumed solution Eq. (3.4.46). Pulling out the $n = 0$ term of the solution and noting that it must vanish leads to Eq. (3.4.47).The process is not too difficult, it's just a lot of algebra I don't want to type out. See, e.g., pp. 266–268 of Ref. 14

Equation (3.4.47) is called the **indicial equation**. The roots r_1 and r_2 determine the nature of the solutions of the ODE:

- If Eq. (3.4.47) has two distinct roots *that do not differ by an integer* (i.e., $r_1 - r_2 \notin \mathbb{Z}$), then we will be able to find two linearly independent solutions to the ODE given by Eq. (3.4.46).
- If the two distinct roots differ by a *nonzero* integer (i.e., $r_1 - r_2 \in \mathbb{Z}_{\neq 0}$), then the solution is given by

$$x(t) = C_1 x_1(t) \ln t + \sum_{n=0}^{\infty} d_n t^{n+r_2}\,, \tag{3.4.50}$$

  where

$$x_1(t) = \sum_{n=0}^{\infty} c_n t^{n+r_1}\,. \tag{3.4.51}$$

  Note that c_0, $d_0 \neq 0$, but that C_1 might be 0. In that case, we are only guaranteed one solution, namely $x_1(t)$.
- If $r_1 - r_2 = 0$, then we will have two linearly independent solutions given by Eq. (3.4.50). That is, if the roots are equal, C_1 will not be 0.

3.4.6 Higher Order ODEs—Particular Solutions

Now that we've seen a few techniques for finding the homogeneous solution to the ODE—which we will always have to do—we can start thinking about how to deal with that pesky right hand side $f(t)$. Note that in all cases, the methods assume that the ODE has put in standard form. That is, the coefficient of the highest derivative must be 1.

Undetermined Coefficients

Suppose our ODE has constant coefficients. Then we can factor the operators, find the roots of the characteristic equation, and form our exponential solutions. For example, for a second order equation

$$\left(D^2 + A\,D + B\right) x = f(t)\,. \tag{3.4.52}$$

the homogeneous solution is

$$x_h = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}\,, \tag{3.4.53}$$

where $\lambda_{1,2}$ are the roots to the characteristic equation. Now if $f(t)$ is of an easily differentiable form, then we could try a trial solution and see if we can find coefficients that will satisfy the ODE. Namely if f is a polynomial, trigonometric function, or exponential (or some linear combination of these), we can use a trial function for our particular solution, and see if we can find coefficients to satisfy the ODE, see Table 3.2 .

Nonhomogeneous Part	**Trial Solution**
$f(t) = \text{const.}$	$x_p = C$
$f(t) = p_n(t)$	$x_p = C_n t^n + C_{n-1} t^{n-1} + \ldots + C_1 t + C_0$
$f(t) = \sin at$	$x_p = C_1 \cos at + C_2 \sin at$
$f(t) = \cos at$	
$f(t) = e^{at}$	$x_p = C_n e^{at}$

Table 3.2: Trial solutions for the undetermined coefficients method for various nonhomogeneous parts; after Table 3.4.1 in Ref. 14.

Note that if $f(t)$ is a sum or product of these forms, than the trial solution will be the corresponding sum or product. For example, suppose we have

$$\left(D^2 + A\,D + B\right) x = t e^{at} \cos bt\,. \tag{3.4.54}$$

Then our trial solution will be

$$x_p = (C_1 + C_2 t) e^{at} \cos bt + (C_3 + C_4 t) e^{at} \cos bt\,. \tag{3.4.55}$$

Trial Solution in Homogeneous Solution

If the trial form x_p *already appears* in the homogeneous solution x_h, then multiply the term by t first. For example, if we want to solve

$$(D-1)(D-2)x = e^{2t}\,, \tag{3.4.56}$$

then $x_p = C_1 e^{2t}$ is clearly already part of $x_h(t)$. In this case, our trial solution should be $x_p = C_1 t e^{2t}$.

Variation of Parameters

Suppose we want to solve the second-order ODE. Rearranging Eq. (3.4.1) with $n = 2$, we have

$$x'' + p(t)x' + q(t)x = f(t)\,. \tag{3.4.57}$$

And let's suppose that we already know a complementary solution to the homogeneous problem

$$\begin{aligned} x_h &= C_1 x_1(t) + C_2 x_2(t)\,, \\ &\text{where} \quad x_h'' + p(t)x_h' + q(t)x_h = 0\,. \end{aligned} \tag{3.4.58}$$

It can be shown that a general solution is[†]

$$x(t) = x_h(t) + u_1(t)x_1 + u_2(t)x_2\,, \tag{3.4.59}$$

where

$$u_1(t) = -\int \frac{x_1\, f}{W(x_1,\, x_2)}\,\mathrm{d}t \tag{3.4.60}$$

$$u_2(t) = \int \frac{x_2\, f}{W(x_1,\, x_2)}\,\mathrm{d}t\,. \tag{3.4.61}$$

Here, $W(x_1, x_2)$ is determinant of the Wronskian

$$W = \begin{vmatrix} x_1(t) & x_2(t) \\ \frac{\mathrm{d}x_1}{\mathrm{d}t} & \frac{\mathrm{d}x_2}{\mathrm{d}t} \end{vmatrix}\,. \tag{3.4.62}$$

Similar results follow for higher order order solutions. For instance, if we have a third-order ODE, then each function u_i (which multiplies the homogeneous solution x_i) obeys

$$u_i' = \frac{W_i}{W\,(x_1,\, x_2,\, x_3)}\,. \tag{3.4.63}$$

[†] This result is not terribly difficult to obtain. Start by assuming a solution of the form

$$y_p = u_1 x_1 + u_2 x_2\,,$$

do a lot of chain rules to find the derivatives, and then cancel out the terms proportional to x_1 and x_2. The assumption is made that $u_1' x_1 = u_2' x_2$ for no reason other than that it's convenient.

In Eq. (3.4.63) W_i is the determinant of the Wronskian with the i^{th} column replaced with $[0, 0, f(t)]^{\mathsf{T}}$. So, e.g., for $i = 2$

$$W_2 = \begin{vmatrix} x_1 & 0 & x_3 \\ x_1' & 0 & x_3' \\ x_1'' & f(t) & x_3'' \end{vmatrix} . \tag{3.4.64}$$

> “*Usually it is not a good idea to memorize formulas in lieu of understanding a procedure. However, the foregoing procedure is too long and complicated to use each time we wish to solve a differential equation. In this case it is more efficient to simply use [Eq.* (3.4.63)*].*”
>
> —pp. 135–136 of Ref. 14

Cramer's Rule It will be useful to review Cramer's rule, since it is useful both when solving variation of parameters problems and when deriving Newton's method in more than 1 dimension (see Sec. 3.6.1). Cramer's rule says that for

$$\underbrace{\begin{pmatrix} x_1 & x_2 \\ x_1' & x_2' \end{pmatrix}}_{\text{Wronskian}} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} ,$$

we can find the solution for u_1 found by taking the determinant of the Wronskian when the first column is replaced by the right hand side and dividing by the determinent of the Wronskian. Similarly, the solution for u_2 is found the same way, replacing the second column in the numerator. That is,

$$u' = \frac{1}{W} \begin{vmatrix} g_1 & x_2 \\ g_2 & x_2' \end{vmatrix} , \quad \text{and} \quad v' = \frac{1}{W} \begin{vmatrix} x_1 & g_1 \\ x_1' & g_2 \end{vmatrix} ,$$

where

$$W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} .$$

The origin of Eqs. (3.4.60) and (3.4.61) can be recalled from these, if we remember that $g_1 = 0$, and $g_2 = f(t)$

3.4.7 Laplace Transforms

The Laplace transform $\hat{f}(s)$ of a function $f(t)$ is defined by

$$\hat{f}(s) = \mathcal{L}[f(t)] \equiv \int_0^\infty f(t)\, e^{-st}\, \mathrm{d}s\,. \tag{3.4.65}$$

It turns out this transform can be useful to solve ODEs with constant coefficients. Suppose we have a spring-mass-damper system (stiffness k, mass m, damping b) that is being forced at $F(t)$. Its equation of motion is

$$m\ddot{x} + b\dot{x} + kx = F(t)\,. \tag{3.4.66}$$

It can be shown from the definition [Eq. (3.4.65)] that

$$\mathcal{L}[f'(t)] = s\hat{f}(s) - f(0), \tag{3.4.67}$$

and by subsequent application

$$\mathcal{L}[f''(t)] = s^2\hat{f}(s) - sf(0) - f'(0)\,. \tag{3.4.68}$$

So, if we take the Laplace transform of both sides of Eq. (3.4.66), we get

$$m\left[s^2\hat{x} - s\cdot x_0 - \dot{x}_0\right] + b\left[s\hat{x} - x_0\right] + k\hat{x} = \hat{F}, \tag{3.4.69}$$

where x_0 and $\dot{x}_0$ are the initial position and velocity, respectively. Let's say the mass is at rest at $x = 0$ to start for simplicity. Then,

$$\begin{aligned} \left(ms^2 + bs + k\right)\hat{x} &= \hat{F} \\ \implies \hat{x} &= \frac{\hat{F}}{ms^2 + bs + k}\,. \end{aligned} \tag{3.4.70}$$

While there is an inverse Laplace transform, it in general requires performance of a contour integral. Since this is outside the scope of the topics required for the exam, it will not be discussed here.[†]

Instead, to find inverse transforms, we will rely on pattern matching techniques. That is, we will try to find $\hat{f}$ in a form where it looks like a combination of known forward transforms. We can then use properties of the Laplace transform (mainly linearity) to find what the original function was.

[†]For more information, see, e.g., Ref. 15, pp. 30–31.

Partial Fraction Decomposition As we've said, we often need to find the inverse Laplace transform using pattern matching. For example, suppose we solve for the Laplace transform of the function and have

$$\hat{f} = \frac{c_1 s^2 + c_2 s + c_3}{d_1 s^3 + d_2 s^2 + d_3 s + d_4}, \tag{3.4.71}$$

for some constants c_n and d_n. This would be a bit of a mess to take the inverse transform of. Instead, suppose we can factor the denominator of Eq. (3.4.71) to write

$$\hat{f} = \frac{c_1 s^2 + c_2 s + c_3}{(s - k_1)(s - k_2)^2}. \tag{3.4.72}$$

Write the right side of this equation as the sum of fractions with each factor as the denominator, and polynomials of one degree less as the numerator. If a factor has multiplicity greater than one, write one fraction for for each power of the factor. So, for Eq. (3.4.72), we would write

$$\frac{c_1 s^2 + c_2 s + c_3}{(s - k_1)(s - k_2)^2} = \frac{A}{s - k_1} + \frac{B}{s - k_2} + \frac{C}{(s - k_2)^2} \tag{3.4.73}$$

[note that both $(s - k_2)$, and $(s - k_2)^2$ appear]. Equation (3.4.73) may then be solved for A, B, C, and D, and the inverse transform can usually found by inspection.

Some common Laplace Transform pairs are given in Table 3.3. Suppose for example that by the above procedure, we found

$$\hat{x}(s) = \frac{12}{s^4} + \frac{1}{3s^2 + 27}. \tag{3.4.74}$$

Rearranging a bit and comparing with the table allows us to write

$$\begin{aligned} \hat{x}(s) &= 2\frac{6}{s^4} + \frac{3}{s^2 + 9} \\ &\implies x(t) = 2t^3 + \sin 3t. \end{aligned} \tag{3.4.75}$$

All past exams have provided a table like Table 3.3 when needed (i.e., I doubt we need to memorize them). I think they are sometimes given when there is a simpler technique in order to throw you off. But of course, this technique can always be used for $t > 0$.

Function $[f(t)]$	**Transform** $[\hat{f}(s)]$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$

Table 3.3: Laplace transform pairs for some common functions. Note that strictly these are true only for $t > 0$.

3.4.8 Solution of Systems of ODEs with Matrix Methods

This technique was mentioned in the context of first-order ODEs, but we'll flesh it out here, since it allows us to solve higher order linear ODEs. The problem we'll solve is

$$\boldsymbol{x}'(t) = A(t)\,\boldsymbol{x}(t), \tag{3.4.76}$$

where

$$\boldsymbol{x} = \Big(x_1(t),\ \ x_2(t),\ \ \cdots,\ \ x_n(t)\Big)^{\mathsf{T}}, \tag{3.4.77}$$

and $A \in \mathbb{R}^{n\times n}$ whose entries may be functions of t but not the functions $x_i(t)$. But wait, we're interested in *higher order* ODEs, whereas Eq. (3.4.76) only has a first derivative. Hm. Well suppose we want to solve

$$x'' + x' + 4x = 0\,. \tag{3.4.78}$$

Let $y = x'$, so that our problem is

$$x' = y \tag{3.4.79}$$

$$y' = -y - 4x\,. \tag{3.4.80}$$

So we've reduced our 2nd order ODE to a system of coupled first order ODEs. We can always do this if the ODE is linear.

In order to actually solve these problems, we'll assume that our matrix A is constant. The problem then reduces to finding the eigenvalues and eigenvectors of A. If A has n linearly independent eigenvectors $\boldsymbol{v}_i$, then

$$\boldsymbol{x} = c_1 e^{\lambda_1 t}\,\boldsymbol{v}_1 + c_2 e^{\lambda_2 t}\,\boldsymbol{v}_2 + \cdots c_n e^{\lambda_n t}\,\boldsymbol{v}_n\,. \tag{3.4.81}$$

Note that Eq. (3.4.81) is valid *even if two eigenvectors correspond to the same eigenvalue.* We require only that the n eigenvectors are linearly independent.

But what if for some eigenvalue λ with multiplicity 2, we find only one eigenvector? In this case we find the first eigenvector as usual

$$(A - \lambda \mathbf{1})\boldsymbol{v}_1 = \mathbf{0}\,. \tag{3.4.82}$$

The second vector, we find from the first

$$(A - \lambda \mathbf{1})\boldsymbol{v}_2 = \boldsymbol{v}_1\,. \tag{3.4.83}$$

The solutions corresponding to this eigenvalue are then

$$c_1\boldsymbol{v}_1\, e^{\lambda t} + c_2\Big(\boldsymbol{v}_1 t e^{\lambda t} + \boldsymbol{v}_2 e^{\lambda t}\Big)\,. \tag{3.4.84}$$

The pattern continues, so that if we had just one eigenvector for an eigenvalue of multiplicity 3, we would have

$$c_1\boldsymbol{v}_1\, e^{\lambda t} + c_2\Big(\boldsymbol{v}_1 t e^{\lambda t} + \boldsymbol{v}_2 e^{\lambda t}\Big) + c_3\left(\boldsymbol{v}_1\frac{t^2}{2}e^{\lambda t} + \boldsymbol{v}_2 t e^{\lambda t} + \boldsymbol{v}_3 e^{\lambda t}\right)\,. \tag{3.4.85}$$

3.5 Linear Partial Differential Equations

3.5.1 Classification of PDEs

Linear partial differential equations (PDEs) of two variables (let's call them x and t) can be written in the form

$$Af_{xx} + Bf_{xt} + Cf_{tt} + Df_x + Ff_t + Gf + H = 0\,, \tag{3.5.1}$$

where $f_x = \partial f/\partial x$. These equations are classified as parabolic, elliptic, or hyperbolic, depending on the values of A, B, and C; see Table 3.4.

Classification	**Characteristic**	**Example**
Parabolic	$B^2 - 4AC = 0$	Heat Equation: $\frac{\partial f}{\partial t} - k\nabla^2 f = 0$
Elliptic	$B^2 - 4AC < 0$	Laplace's Equation: $\nabla^2 f = 0$
Hyperbolic	$B^2 - 4AC > 0$	Wave Equation: $\frac{\partial^2 f}{\partial t^2} - c^2\nabla^2 f = 0$

Table 3.4: Classification of linear PDEs

3.5.2 Separation of Variables

A general approach for PDEs is to assume that the solution $u(x, y, z, t)$ can be be written as a product of functions of a single variable only; that is we assume

$$u(x, y, z, t) = X(x)Y(y)Z(z)T(t). \tag{3.5.2}$$

Substitution of the separated form into the governing PDE allows generation of a set of ODEs (one for each variable) that are easier to solve. The solutions to these individual ODEs are coupled by the separation constant, whose value is determined by the initial and boundary conditions.

Consider the 1D heat equation, which says that the temperature u obeys

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}, \tag{3.5.3}$$

where α is a constant.[†] If we assume a separable solution [Eq. (3.5.2)], then Eq. (3.5.3) becomes

$$XT' = \alpha X''T, \tag{3.5.4}$$

where the primes denote differentiation with respect to the function's argument. Rearranging, we have

$$\frac{T'}{T} = \alpha \frac{X''}{X}. \tag{3.5.5}$$

Note that the left side of Eq. (3.5.5) depends only on time, while the right side depends only on position x. Therefore, Eq. (3.5.5) can only be true if both sides are equal to a constant (say, λ). We can then write two ODEs

$$\frac{\partial T}{\partial t} - \lambda T = 0 \tag{3.5.6}$$

$$\frac{\partial^2 X}{\partial x^2} - \frac{\lambda}{\alpha} X = 0. \tag{3.5.7}$$

Solutions can to the ODEs Eqs. (3.5.6) and (3.5.7) can be found in order to satisfy the boundary conditions with the methods of Sec. 3.4.

Now in general it's convenient to write the separation constant as a squared value, e.g., $\lambda = \pm\beta^2$. The choice of sign will depend on the boundary conditions, but it is usually best to choose a *negative* separation constant $\lambda = -\beta^2$ for the wave and heat equations.

[†]Namely, $\alpha = k/c_p$ is the thermal diffusivity, where k is the material's thermal conductivity, and c_p is its heat capacity.

Boundary Conditions

Typical boundary conditions for the heat and wave equations correspond to intuitive physical conditions. Usually these translate simply to the requirement that the field or the derivative vanish at the boundary.[†] Some typical conditions for the 1D wave and heat equations are given in Table 3.5.

Equation	Physical Condition	Boundary Condition
Heat Equation	Constant Temperature	$u = \text{const.}$
	Insulated	$u_x = 0$
Wave Equation	Rigid	$u_x = 0$
	Free	$u = 0$

Table 3.5: Boundary Conditions for prototype equations

Initial Conditions

Separation of variables over a finite domain will usually yield a series solution, e.g., for the heat equation when the temperature is always 0 at both boundaries,

$$u(x,t) = \sum_{n=0}^{\infty} C_n \sin\frac{n\pi x}{L}\; e^{-k\left(\frac{n\pi}{L}\right)^2 t}\,. \tag{3.5.8}$$

In order to determine what those C_ns should be, we need the initial condition. Usually this is given as $u(x,0) = f(x)$. Substituting $t = 0$ into Eq. (3.5.8) gives

$$f(x) = \sum_{n=0}^{\infty} C_n \sin\frac{n\pi x}{L}\,. \tag{3.5.9}$$

Now if we multiply both sides by $\sin m\pi x/L$ and integrate from 0 to L, we have

$$\int_0^L f(x)\,\sin\frac{m\pi x}{L}\,\mathrm{d}x = \int_0^L \left(\sum_{n=0}^{\infty} C_n \sin\frac{m\pi x}{L}\,\sin\frac{n\pi x}{L}\right)\mathrm{d}x\,. \tag{3.5.10}$$

[†] Requirement that the function attain a certain value at the boundary is called a Diriclet condition. If we force the derivative of the function to a certain value at the boundary, this is a Neumann condition. Something which has both, e.g., if the boundary condition is

$$u_x + 3u = 0 \quad \text{at} \quad x = L\,,$$

is termed a mixed boundary condition.

But since the trigonometric functions are orthogonal on $(0, L)$, we see that the integral will vanish for all terms *except* when $n = m$. Thus only the single $m = n$ term remains from the summation and

$$\begin{aligned}\int_0^L f(x)\,\sin\frac{m\pi x}{L}\,\mathrm{d}x &= C_m\int_0^L \sin^2\frac{m\pi x}{L}\,\mathrm{d}x\\ &= C_m(L/2)\\ \implies C_n &= \frac{2}{L}\int_0^L f(x)\,\sin\frac{n\pi x}{L}\,\mathrm{d}x\,. \end{aligned}\tag{3.5.11}$$

Note that the orthogonality condition holds when the argument of the trigonometric functions is augmented by half integers. For instance, if we have Dirichlet boundary condition (u must vanish) on one end and a Neumann condition (u_x must vanish) on the other, we'll have a series solution that looks like

$$u(x,t) = \sum_{n=0}^{\infty} C_n \cos\frac{(n+1/2)\pi x}{L}\; e^{-k(\frac{n\pi}{L})^2 t}\,. \tag{3.5.12}$$

The orthogonality exploited over $(0, L)$ holds in this case as well.

Nonhomogeneous Boundary Conditions

If the boundary conditions are that the function or its derivative vanish at the end points, the solution of the PDE by separation of variables is pretty routine. But what if we have nonvanishing—or even worse, *time-dependent*—boundary conditions? Well, this is a bit trickier, but can be handled. We'll look at a few examples for the heat equation.

Time-Independent BCs We'll first consider the unforced case where the temperature u must equal a different *constant* at each end of the domain:

$$u_{xx} = u_t \qquad u(0,t) = u_0\,,\quad u(L,t) = u_1\,,\quad u(x,0) = f(x)\,. \tag{3.5.13}$$

Since the boundary conditions impose a spatial constraint for all time, we try writing

$$u(x,t) = v(x,t) + \phi(x)\,, \tag{3.5.14}$$

where v is a solution of the problem with homogeneous boundary conditions. Substitution of Eq. (3.5.14) into Eq. (3.5.13) gives

$$v_{xx} + \phi_{xx} = v_t\,. \tag{3.5.15}$$

But since $v_{xx} = v_t$, and since v vanishes at the endpoints, our problem is reduced to and ODE for ϕ:

1. $\phi'' = 0$,
2. $\phi(0) = u_0$,
3. $\phi(L) = u_0$.

Then the first condition requires

$$\phi = Ax + B\,. \tag{3.5.16}$$

The second condition together with Eq. (3.5.16) means that

$$\phi(0) = A(0) + B = u_0 \rightarrow B = u_0\,. \tag{3.5.17}$$

Finally, the last condition means that

$$\phi(L) = A(L) + u_0 = u_1 \rightarrow A = \frac{1}{L}\left(u_1 - u_0\right)\,. \tag{3.5.18}$$

Substituting these back into Eq. (3.5.16) gives

$$\phi(x) = u_0 + \frac{x}{L}\left(u_1 - u_0\right)\,. \tag{3.5.19}$$

We can now solve the much easier problem

$$v_{xx} = v_t \qquad v(0,t) = v(L,t) = 0\,. \tag{3.5.20}$$

Note, however, that the initial condition has changed! Since the problem specified $u(x) = f(x)$ at $t = 0$, we must have

$$v(0,x) = u(0,x) - \phi(x)\,. \tag{3.5.21}$$

This technique also works if there is a nonhomogeneous part of the PDE. That is, suppose we had a forcing term in Eq. (3.5.13)

$$u_{xx} + F(x) = u_t\,. \tag{3.5.22}$$

We again try to find a solution in the form of Eq. (3.5.14)

$$u(x,t) = v(x,t) + \phi(x)\,, \tag{3.5.23}$$

but instead of requiring that ϕ_{xx} vanish, we demand that

$$\phi_{xx} + F(x) = 0\,. \tag{3.5.24}$$

Time-Dependent BCs Suppose we again need to solve the 1D heat equation, but the temperature at the boundaries $x = 0$ and $x = L$ is a specified function of time:

$$u_{xx} + F(x,t) = u_t\,, \qquad u(0,t) = u_0(t)\,, \qquad u(L,t) = u_1(t)\,. \tag{3.5.25}$$

As before we'll seek a solution that is the sum of the homogeneous problem, and a solution that satisfies the boundary conditions:

$$u = v(x,t) + \phi(x,t)\,. \tag{3.5.26}$$

It turns out that Eq. (3.5.16) can be extended to the time-dependent case pretty naïvely:

$$\phi(x,t) = u_0(t) + \frac{x}{L}\left[u_1(t) - u_0(t)\right]\,. \tag{3.5.27}$$

Thus ϕ will satisfy the boundary conditions. Now, we have to solve the homogeneous problem

$$k v_{xx} + \underbrace{\left[F(x,t) - \phi(x,t)\right]}_{G(x,t)} = v_t\,. \tag{3.5.28}$$

Equation (3.5.28) doesn't really seem any easier to solve—we still have that forcing function $F(x,t) - \phi(x,t)$. However, we now have a problem that has homogeneous boundary conditions! Thus, we can expand in terms of the basis functions

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin\frac{n\pi x}{L}\,, \tag{3.5.29}$$

$$G(x,t) = \sum_{n=1}^{\infty} G_n(t) \sin\frac{n\pi x}{L}\,. \tag{3.5.30}$$

The function $G(x,t) = F(x,t) - \phi(x,t)$ is known, since $F(x,t)$ is given, and we found $\phi(x,t)$ from Eq. (3.5.27). The final step is to substitute Eqs. (3.5.29) and (3.5.30) into Eq. (3.5.28) and match powers of $\sin(n\pi x/L)$. This will give a first order ODE for v_n that shouldn't be too terrible to solve. Once we have the v_ns, we have all the terms in Eq. (3.5.26). All that remains is to apply the boundary condition in the usual way (see **Initial Conditions**).

3.5.3 Integral Transform Methods

While separation of variables is the usual way to approach PDEs, if the domain is infinite or semi-infinite, we'll have a much harder time matching the "boundary conditions" (since the domain is infinite, usually this condition is simply that $u \to 0$ as $x \to \pm\infty$). In this situation, it's often useful to employ an integral transform to make the problem more tractable.

Laplace Transforms

As in the case of ODEs (see Sec. 3.4.7), Laplace transforms can also be used to find solutions to PDEs. Recall Eq. (3.4.65), that the Laplace transform of a function $f(t)$ is defined by

$$\hat{f}(x,s) = \mathcal{L}\left[f(x,t)\right] \equiv \int_0^\infty f(t)\, e^{-st}\, \mathrm{d}t\,. \tag{3.5.31}$$

Note that our derivative theorems hold for t, namely

$$\mathcal{L}\left[\frac{\partial f}{\partial t}\right] = s\hat{f}(s,x) + f(0,x)\,, \tag{3.5.32}$$

while for x,

$$\mathcal{L}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial}{\partial x}\left[\hat{f}(s,x)\right]\,. \tag{3.5.33}$$

Suppose we have a PDE of the form

$$u_x + u_t = F(x)\,. \tag{3.5.34}$$

Taking the Laplace transform of Eq. (3.5.34) gives

$$\hat{u}_x + s\hat{u} + u_0(x) = \hat{F}(x)\,. \tag{3.5.35}$$

We've thus reduced our PDE to an ODE for $\hat{u}(x,s)$, which can usually be solved with the methods in Sec. 3.4.

Fourier Transforms

The temporal Fourier transform of a function $f(t)$ is defined by[†]

$$\tilde{f}(\omega) = \mathcal{F}\left[f(t)\right] \equiv \int_{-\infty}^\infty f(t)\, e^{i\omega t}\, \mathrm{d}t\,, \tag{3.5.36}$$

[†]The sign conventions and location of the scaling factor $1/2\pi$ vary considerably. We'll use the convention that the forward transform uses $+i\omega t$, and that the factor of $1/2\pi$ appears in the inverse operation.

and the inverse transform by

$$\mathcal{F}^{-1}\left[\tilde{f}(\omega)\right] \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t}\, \mathrm{d}\omega\,. \tag{3.5.37}$$

We could also define the spatial transform which maps $x \to \xi$:

$$\tilde{f}(\xi) = \mathcal{F}\left[f(t)\right] \equiv \int_{-\infty}^{\infty} f(x)\, e^{i\xi x}\, \mathrm{d}\xi\,, \tag{3.5.38}$$

with the analogous inverse transform.

The Fourier transform is sometimes equivalently useful for reducing PDEs to ODEs. For instance, suppose we want to solve the heat equation

$$u_t = k u_{xx}\,, \tag{3.5.39}$$

subject to the initial condition $u(x,0) = u_0(x)$. If we take the spatial Fourier transform of Eq. (3.5.39), we have

$$\tilde{u}_t = -k\xi^2 \tilde{u}, \tag{3.5.40}$$

since $\mathcal{F}[\partial_x u(x)] \to i\xi\tilde{u}(\xi)$.

To show that derivatives in the time (or spatial) domain reduce multiplications in the frequency domain, integrate by parts:

$$\begin{aligned}\mathcal{F}\left[f'(t)\right] &= \int_{-\infty}^{\infty} f'(t)\, e^{i\omega t}\, \mathrm{d}t \\ &= i\omega\, f(t)\,\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} i\omega f(t)\, e^{i\omega t}\, \mathrm{d}t\,.\end{aligned} \tag{3.5.41}$$

The first term in Eq. (3.5.41) must vanish, since for the Fourier transform to exist, $f(t)$ must be square integrable, and thus must tend to 0 as $t \to \pm\infty$. Then the second term is recognized as a multiple of $\tilde{f}(\omega)$

$$\begin{aligned}\mathcal{F}\left[f'(t)\right] &= -i\omega \underbrace{\int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, \mathrm{d}t}_{\mathcal{F}[f(t)]} \\ &= -i\omega \tilde{f}(\omega)\,.\end{aligned} \tag{3.5.42}$$

Initial Conditions

Note that it may be best to match the boundary conditions *in the transformed domain*. That is, sometimes requiring that

$$\hat{u}(s,0) = \hat{f}(s), \tag{3.5.43}$$

where $\hat{f} = \mathcal{L}[f(x)]$ is the transform of the initial condition $u(x,0) = f(x)$.

3.6 Elementary Numerical Analysis

This topic kind of epitomizes the exam. I think we can all agree that knowing the rudiments of these techniques is absolutely important, given the prevalence of numerical techniques in many fields. But this test demands not only an understanding of the principles, but memorization of the minutiae. So good luck, because you may just have to pull the coefficients for Gaussian quadrature out your butt.[†]

3.6.1 Root-Finding Techniques

Bisection Method

The midpoint (or "bisection") technique hinges on the observation if a continuous function changes sign between, say, $x = a$ and $x = b$, then (by the intermediate value theorem) there must exist some $c \in (a,b)$ such that $f(c) = 0$. The bisection method repeatedly subdivides the interval to narrow down where this root occurs. The steps for each iteration are

1. Define the midpoint $x = c = (a+b)/2$.
2. Evaluate the function at $x = c$.
   (a) If $f(c) = 0$, then we got lucky and found the root.
   (b) If $f(c)$ and $f(a)$ have the opposite signs, then the root must be in $x \in (a,c)$.
   (c) If $f(c)$ and $f(a)$ have the same sign, then the root must be in $x \in (c,b)$.
3. Set the new interval ($a = a$, $b = c$ for item 2b or $a = c$, $b = b$ for item 2c) and repeat until $|b-a| < \epsilon$.

[†]See p. 124.

False Point Method

The bisection method works reasonably well, but can be slow to converge if the root is much closer to one of the initial endpoints than to the other. After all, we're just blindly taking the midpoint each time rather than using any information about how far the function $f(x)$ is from 0 at either end. To do a bit more, note that we can write the slope of the line connecting the two end points a and b as

$$\text{slope} = \frac{f(b) - f(a)}{b - a}\,. \tag{3.6.1}$$

Now call the point where the line connecting the two endpoints crosses the x axis by c. Then, if we set $f(c) = 0$, we can also write

$$\text{slope} = \frac{f(b) - f(c)}{b - c} = \frac{f(b)}{b - c}\,. \tag{3.6.2}$$

Then from Eqs. (3.6.1) and (3.6.2), we have

$$\begin{aligned} \frac{f(b) - f(a)}{b - a} &= \frac{f(b)}{b - c} \\ \implies c &= b - f(b)\,\frac{b - a}{f(b) - f(a)}\,. \end{aligned} \tag{3.6.3}$$

Then, the steps from the midpoint method are employed (see above).

Newton-Raphson

Consider some function $y = f(x)$ that is differentiable in the neighborhood of the root of interest. Suppose we pick a point x_n somewhere near the root x_0.[†] The value of the function is approximately

$$y \simeq f(x_n) + f'(x_n) \cdot (x - x_n)\,. \tag{3.6.4}$$

The root (call it x_{n+1}) of Eq. (3.6.4) is easy to find; we just set $y = 0$, and then

$$\begin{aligned} f(x_n) + f'(x_n) \cdot (x_{n+1} - x_n) &= 0 \\ \implies x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)}\,. \end{aligned} \tag{3.6.5}$$

[†]This of course assumes we know approximately where the root is. If x_0 is root of $f'(x)$, or if $f(x)$ has several roots, we may need to be careful about this. Convergence was only asked about in a qualitative sense, on one exam to my knowledge; see the section on **Convergence** below. For a more thorough review, see, e.g., Sections 4.6–4.7 of Ref. 16.

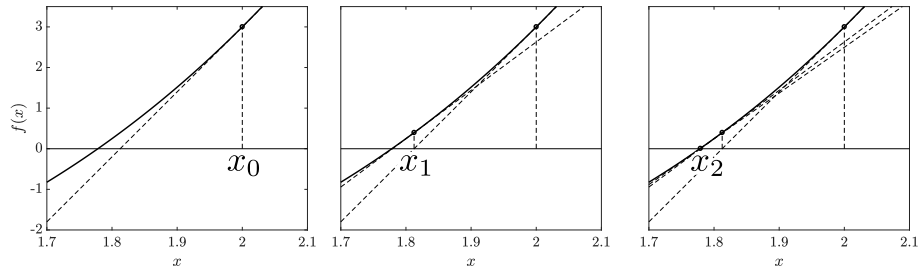


Figure 3.6: First three iterations for Newton's method with an initial guess of $x_0 = 2$, showing rapid convergence to the root.

Now the argument is that the tangent line [Eq. (3.6.4)] will cross the x-axis at x_{n+1} close to the desired root x_0. If we repeat the process using $x = x_{n+1}$ as our guess, we see that x_{n+2} will be even closer to x_0. This process can be repeated until the result converges (i.e., until $|x_{n+1} - x_n| \to 0$). The first three steps of this process are illustrated in Fig. 3.6.

Two Dimensions Newton's method is fairly amenable to extension into higher dimensions (we'll only consider two in this bit, but the process would be similar for more). Suppose we want to solve a system of equations

$$\begin{aligned} f(x, y) &= 0 \\ g(x, y) &= 0\,. \end{aligned} \tag{3.6.6}$$

We can expand in the same way as Eq. (3.6.4), now accounting for the fact that the functions vary in two dimensions

$$\begin{aligned} f(x_{n+1}, y_{n+1}) &\simeq f_n + \frac{\partial f}{\partial x}\,(x_{n+1} - x_n) + \frac{\partial f}{\partial y}\,(y_{n+1} - y_n) \\ g(x_{n+1}, y_{n+1}) &\simeq g_n + \frac{\partial g}{\partial x}\,(x_{n+1} - x_n) + \frac{\partial g}{\partial y}\,(y_{n+1} - y_n)\,, \end{aligned} \tag{3.6.7}$$

where the derivatives are evaluated at the point (x_n, y_n). Since we assume the iterated point is near the root, set $f(x_{n+1}, y_{n+1}) = g(x_{n+1}, y_{n+1}) = 0$, and rearrange Eq. (3.6.7) to find

$$\begin{aligned} f_x\, x_{n+1} + f_y\, y_{n+1} &= -f_n + f_x\, x_n + f_y\, y_n \\ g_x\, x_{n+1} + g_y\, y_{n+1} &= -g_n + g_x\, x_n + g_y\, y_n\,, \end{aligned} \tag{3.6.8}$$

Using Cramer's rule (see the note on p. 99) to find

$$\begin{aligned} x_{n+1} &= x_n - \frac{f_n\, g_y - g_n\, f_y}{f_x\, g_y - g_x\, f_y} \\ y_{n+1} &= y_n - \frac{g_n\, f_x - f_n\, g_x}{f_x\, g_y - g_x\, f_y}\,. \end{aligned} \tag{3.6.9}$$

Secant Method

The secant method to find the root of a function is nearly identical to Newton's method, except that we won't use the analytical derivative to compute the slope of the function. Instead, we'll use a finite difference approximation, namely that

$$f'(x) \simeq \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{3.6.10}$$

Then, subsequent iterated points are given by

$$x_{n+1} = x_n - f(x_n) \Bigg/ \left[\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} \right] \tag{3.6.11}$$

Note that the secant method requires two initial values, so that we can compute the finite difference slope.

Fixed Point Method

For the fixed point method, the equation whose root is to be found is written in the form

$$x = g(x). \tag{3.6.12}$$

We then iterate as

$$x_{n+1} = g(x_n) \tag{3.6.13}$$

Expanding this function around the root x_0, we have

$$g(x) \simeq g(x_0) + g'(x_0)(x - x_0) + \ldots. \tag{3.6.14}$$

But since $g(x_0) = x_0$,

$$g(x) - x_0 = g'(x_0)(x - x_0). \tag{3.6.15}$$

Since we are repeatedly evaluating g to find the root, we iterate as

$$x_{n+1} = g(x_n). \tag{3.6.16}$$

So then from Eq. (3.6.15),

$$x_{n+1} - x_0 = g'(x_0)(x_n - x_0). \tag{3.6.17}$$

Equation (3.6.17) suggests that if $|g'| > 1$, we will multiply the distance between the root and the guess by successively larger amounts and the solution will diverge. Thus when we write $g(x)$ so as to have the form of Eq. (3.6.12), we need to be sure that the magnitude of its derivative is less than unity.[†]

Convergence

It is sometimes useful to know how fast a method will converge to a root. Call the error between the root estimation after k steps (x_k) and the actual root (x_0) by ϵ, that is

$$\epsilon_k = x_k - x_0\,. \tag{3.6.18}$$

We'll define a rate of convergence α so that the error between steps varies as

$$|\epsilon_{k+1}| = C|\epsilon_k|^\alpha\,. \tag{3.6.19}$$

The true root is at x_0, so we can write

$$\begin{aligned} 0 = f(x_0) &= f(x_k + \epsilon_k) \\ &\approx f(x_k) + f'(x_k)\epsilon_k + \frac{1}{2}f''(x_k)\epsilon_k^2 + \dots \\ \implies -\frac{f(x_k)}{f'(x_k)} &= \epsilon_k + \frac{f''(x_k)}{2f'(x_k)}\epsilon_k^2 + \mathcal{O}\left(\epsilon_k^3\right)\,. \end{aligned} \tag{3.6.20}$$

Note that Eq. (3.6.20) is valid when $f'(x_k) \neq 0$. This makes sense; if we chose a point where the derivative of the funtion vanished, we'd have a tangent with 0 slope—which would never cross the x axis.

For example, consider Newton's method. The error from successive iterations will be

$$\begin{aligned} \epsilon_{k+1} &= x_0 - x_{k+1} \\ &= x_0 - \left[x_k - \frac{f(x_x)}{f'(x_k)}\right] \end{aligned} \tag{3.6.21}$$

[†]Derivation follows notes from METRIC, a resource from Imperial College London.

Then, since $x_0 - x_k = \epsilon_k$, and using Eq. (3.6.20), we have from Eq. (3.6.21)

$$\epsilon_{k+1} = \epsilon_k - \left[\epsilon_k + \frac{f''(x_x)}{2f'(x_k)}\epsilon_k^2 + \mathcal{O}\left(\epsilon_k^3\right)\right] \tag{3.6.22}$$

$$= -\frac{f''(x_x)}{2f'(x_k)}\epsilon_k^2 + \mathcal{O}\left(\epsilon_k^3\right) . \tag{3.6.23}$$

Comparing Eq. (3.6.23) with Eq. (3.6.19), we see that

$$C = -\frac{f''(x_x)}{2f'(x_k)}, \qquad \text{with} \qquad \alpha = 2 . \tag{3.6.24}$$

Because the exponent $\alpha = 2$, the method is said to have *quadratic convergence.*[†]

3.6.2 Curve Fitting – Least Squares

Suppose we have some set of points x_n at which we measure some value y_n. We want to find the best-fit polynomial of degree m which through them (as best we can). The best fit curve will be given by

$$f(x_n) = \beta_0 + \beta_1 x_n + \cdots + \beta_m x_n^m . \tag{3.6.25}$$

As a matrix equation, Eq. (3.6.25)

$$f(\boldsymbol{x}) = \beta_0 + \beta_1 \boldsymbol{x} + \cdots + \beta_m \boldsymbol{x}^m . \tag{3.6.26}$$

But, as long as there are more than m points, Eq. (3.6.26) is overdetermined—that is, there is no exact solution unless the points *are* the points of a polynomial (in which case there's little point in fitting a curve). Instead, we compute ϵ, the total error between value of the curve at each x_n and measured value, i.e., $\|y_n - f(x_n)\|$, and attempt to minimize it. Summing over all points, the error is[‡]

$$\epsilon = \sum_{n=1}^{N} \left[y_n - f(x_n)\right]^2 . \tag{3.6.27}$$

We now try to find the values of the coeffcients β_m such that ϵ is minimized. Since this is an extreme value, we compute the partial derivatives and set them

[†]Presentation is due to Mark Arnold, notes from numerical analysis course, fall 2016.

[‡]We'll minimize the square instead of the absolute value, since their minima will be the same and the exponent is more tractable. For this reason, the error $\|y_n - f(x_n)\|$ is sometimes called the "residual", and ϵ termed the "r^2 value".

equal to 0:

$$\begin{aligned}\frac{\partial \epsilon}{\partial \beta_m} &= \frac{\partial}{\partial \beta_m}\sum_{n=1}^{N}\left[y_n - f(x_n)\right]^2 \\ &= -2\sum_{n=1}^{N}\left[y_n - f(x_n)\right]\frac{\partial f}{\partial \beta_m} \\ \implies &\sum_{n=1}^{N}\left[y_n - f(x_n)\right]\frac{\partial f}{\partial \beta_m} = 0 \end{aligned} \tag{3.6.28}$$

The $m+1$ equations with given by Eq. (3.6.28) are a system of linear equations.

As an example, suppose we want to fit three points with a line. Then, $N = 3$, $m = 1$, our best fit line has the form $f(x_n) = \beta_0 + \beta_1 x$. The equations to be solved are

$$\sum_{n=1}^{3}\left[y_n - \beta_0 - \beta_1 x_n)\right] = 0\,, \quad \text{and} \tag{3.6.29}$$

$$\sum_{n=1}^{3}\left[y_n - \beta_0 - \beta_1 x_n)\right] x_n = 0\,. \tag{3.6.30}$$

Evaluation of Eqs. (3.6.29) and (3.6.30) will yeild two equations of the form

$$\begin{aligned} c_1\beta_0 + c_2\beta_1 &= 0 \\ c_3\beta_0 + c_4\beta_1 &= 0 \\ \implies \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}&\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \mathbf{0}, \end{aligned} \tag{3.6.31}$$

where the c_n are constants. Equation (3.6.31) may be solved with the usual methods.

3.6.3 Functional Approximation

Fourier Series

Any square integrable function f which is periodic on some interval $(-L, L)$ can be written as a sum of sinusoidal functions

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\,, \tag{3.6.32}$$

where

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,\mathrm{d}x \\ &\implies a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,\mathrm{d}x\,. \end{aligned} \tag{3.6.33}$$

Polynomial Series

An continuously differentiable function can be approximated in the neighborhood of a point x_0 by its Taylor series:

$$\begin{aligned} f(x_0+\Delta x) &\simeq f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2!}(\Delta x)^2 + \frac{f'''(x_0)}{3!}(\Delta x)^3 + \ldots \\ &\simeq \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(\Delta x)^n\,. \end{aligned} \tag{3.6.34}$$

3.6.4 Finite Difference Methods

The finite difference method is a technique to dealing with differential equations that uses finite intervals to approximate analytical derivatives. For example, consider function f, whose value is known at some point x_0. We can perform a Taylor expansion of the function about x_0 that[†]

$$f(x_0+\Delta x) \simeq f(x_0) + f'(x_0)\Delta x + \frac{f''(x_0)}{2}(\Delta x)^2 + \frac{f'''(x_0)}{6}(\Delta x)^3 + \ldots \tag{3.6.35}$$

$$f(x_0-\Delta x) \simeq f(x_0) - f'(x_0)\Delta x + \frac{f''(x_0)}{2}(\Delta x)^2 - \frac{f'''(x_0)}{6}(\Delta x)^3 + \ldots\,. \tag{3.6.36}$$

Now subtract Eq. (3.6.35) from Eq. (3.6.36) and discard cubic and higher terms to obtain

$$\frac{\mathrm{d}f}{\mathrm{d}x} \simeq \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x}. \tag{3.6.37}$$

Alternatively, adding Eqs. (3.6.35) and (3.6.36) gives

$$f(x_0+\Delta x) + f(x_0-\Delta x) \simeq 2f(x_0) + \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(\Delta x)^2 + \mathcal{O}\left[(\Delta x)^4\right], \tag{3.6.38}$$

[†]See pp. 6–9 of Ref. 17.

or, rearranging, that

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \simeq \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{(\Delta x)^2}. \tag{3.6.39}$$

Note that Eqs. (3.6.37) and (3.6.39) are expressions for the first and second derivatives that use only values of the function itself. These relations neglect terms proportional to $(\Delta x)^2$ and higher; see **Error Analysis** below.

The above approximation relies on knowing the function value at three points: x_0, $x_0 - \Delta x$, and $x_0 + \Delta x$. We could also approximate the derivative by simply rearranging Eq. (3.6.35) and neglecting terms that are quadratic or higher in $(\Delta x)^2$

$$\frac{\mathrm{d}f}{\mathrm{d}x} \simeq \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{3.6.40}$$

The same procedure with Eq. (3.6.36) yields

$$\frac{\mathrm{d}f}{\mathrm{d}x} \simeq \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x}. \tag{3.6.41}$$

For reasons that are probably pretty clear, Eq. (3.6.37) is called the central difference approximation, while Eqs. (3.6.40) and (3.6.41) are termed the forward and backward difference approximations, respectively.

Error Analysis

Note that if we retained more terms in the difference between Eqs. (3.6.35) and (3.6.36), we would find

$$f(x_0 + \Delta x) - f(x_0 - \Delta x) \simeq 2f'(x_0)\Delta x + \frac{f'''(x_0)}{6}(\Delta x)^3 + \frac{f^{(\mathrm{v})}(x_0)}{120}(\Delta x)^5 + \ldots \tag{3.6.42}$$

Dividing Eq. (3.6.42) by $2\Delta x$ and rearranging, we have

$$\begin{aligned} \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} &\simeq f'(x_0) + \frac{f'''(x_0)}{12}(\Delta x)^2 + \frac{f^{(\mathrm{v})}(x_0)}{240}(\Delta x)^4 + \ldots \\ &\simeq f'(x_0) + \mathcal{O}\left[(\Delta x)^2\right]. \end{aligned} \tag{3.6.43}$$

For this reason, the central difference method is said to be accurate to second order in the step size Δx. A similar analysis for the forward and backward difference approxmiations [Eqs. (3.6.40) and (3.6.41)] show that these methods are accurate to first order in the step size. These results are summarized in Table 3.6.

Method	Approximation	Error Bound
Central Difference	$\frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}$	$(\Delta x)^2$
Backward Difference	$\frac{f(x_0)-f(x_0-\Delta x)}{\Delta x}$	Δx
Forward Difference	$\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$	Δx

Table 3.6: Order of the error in $\mathrm{d}f/\mathrm{d}x$ for finite difference approximations with step size Δx.

Solving PDEs—Explicit Methods

Perhaps the most straightforward way to solve PDEs using the finite difference approximations to derivatives are explicit methods. That is, suppose we apply our approximations to the 1D heat equation

$$u_t = k u_{xx} \,. \tag{3.6.44}$$

Substitution of Eqs. (3.6.39) and (3.6.40) into Eq. (3.6.44) and rearrangement leads to

$$u_m^{n+1} = u_m^n + \beta \left(u_{m+1}^n + 2u_m^n + u_{m-1}^n \right) , \tag{3.6.45}$$

where $\beta \equiv k\Delta t/(\Delta x)^2$, and $u_m^n = u(m\Delta x, n\Delta t)$. So, given the initial temperature distribution $u(x, 0)$ and boundary conditions, we can find the temperature at all subsequent times with Eq. (3.6.45).

Stability Obviously we need to ensure that the step sizes in time and space chosen are appropriate so that the solution is stable.[†] The derivation is not terribly difficult, but its details are probably outside the scope of the exam.[‡] It's likely more useful to memorize the results, especially for the heat equation (which is the only one they seem to ask about).

The basic steps in the derivation are

1. Define the error $\epsilon_n^m = u_n^m - f(x_m, t_n)$ as the difference between the true solution f and the finite difference solution u.

[†]Note that the stability depends on the scheme, not the equations.

[‡]A good review is given by Peter J. Olver in Chapter 11 of his Numerical Analysis Lecture Notes (2008)

2. Note that since $f(x_m, t_n)$ is a solution of the governing equation, and thus by linearity, so too is ϵ_n^m.

3. Represent ϵ as a Fourier series, and note that each frequency component must satisfy the governing equation (since it's linear). Then we only need to analyze one frequency.

4. Perform substitutions and cancellations, and define an amplification factor $G = \|\epsilon_{n+1}^m / \epsilon_n^m\|$. Stability requires $G < 1$.

Results are listed in Table 3.7. Note that these results assume the indicated bounds on accuracy of the finite difference scheme (see the bit about Error Analysis in Sec. 3.6.4).

Equation	**Stability Condition**	**Approximation Error**
Heat Equation	$\left\|\frac{k\Delta t}{(\Delta x)^2}\right\| \leq \frac{1}{2}$	$\mathcal{O}\left[\Delta t + (\Delta x)^2\right]$
Wave Equation	$\|\frac{c\Delta t}{\Delta x}\| \leq 1$	$\mathcal{O}\left[(\Delta t)^2 + (\Delta x)^2\right]$

Table 3.7: Stability conditions for step sizes for various ODEs. Note that these conditions are for *forward difference* methods; backward difference (implicit) methods are always stable.

Solving PDEs—Implicit Methods

Consider again the 1d heat equation Eq. (3.6.46)

$$u_t = k u_{xx} \,. \tag{3.6.46}$$

What if, instead of evaluating the time derivative with a forward difference approximation, we use a backward difference method? In this case, substitution of the derivative approxmiations and rearrangement yield

$$u_m^n = -\beta u_{m-1}^{n+1} + (1 + 2\beta)\, u_m^{n+1} - \beta u_{m+1}^{n+1} \,. \tag{3.6.47}$$

But now, we need to know the the values of u at the $(n+1)^{\text{th}}$ step—but how?

Well, since we must have an initial distribution u_0, we can set up a matrix equation to find these values. As an illustration of the process, suppose we have a 1D rod and divide it into four sections. The rod initially has temperature 0,

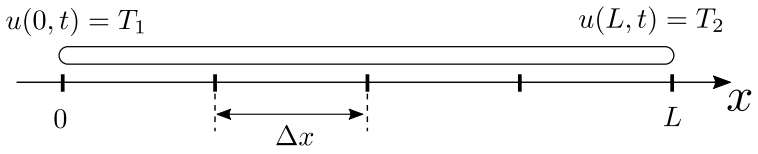


Figure 3.7: One-dimensional bar with fixed end temperatures.

and its ends are held at T_1 and T_2, see Fig. 3.7.

Consider the first interior node ($m = 1$), for which we can write

$$u_1^0 = -\beta u_0^1 + (1 + 2\beta)u_1^1 - \beta u_2^1 . \tag{3.6.48}$$

And similarly for the subsequent nodes,

$$u_2^0 = -\beta u_1^1 + (1 + 2\beta)u_2^1 - \beta u_3^1 \tag{3.6.49}$$

$$u_3^0 = -\beta u_2^1 + (1 + 2\beta)u_3^1 - \beta u_4^1 . \tag{3.6.50}$$

The boundary conditions mean that $u_0^n = T_1$ and $u_4^n = T_2$. So from Eq. (3.6.48)

$$\begin{aligned} u_1^0 &= -\beta T_1 + (1 + 2\beta)u_1^1 - \beta u_2^1 \\ &\implies u_1^0 + \beta T_1 = (1 + 2\beta)u_1^1 - \beta u_2^1 , \end{aligned} \tag{3.6.51}$$

and from Eq. (3.6.50),

$$\begin{aligned} u_3^0 &= -\beta u_2^1 + (1 + 2\beta)u_3^1 - \beta u_4^1 \\ &\implies u_3^0 + \beta T_2 = -\beta u_2^1 + (1 + 2\beta)u_3^1 . \end{aligned} \tag{3.6.52}$$

We can then use Eqs. (3.6.49), (3.6.51) and (3.6.52) to write a matrix equation

$$\begin{pmatrix} 1 + 2\beta & -\beta & \\ -\beta & 1 + 2\beta & -\beta \\ & -\beta & 1 + 2\beta \end{pmatrix} \begin{pmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{pmatrix} = \begin{pmatrix} u_1^0 + \beta T_1 \\ 0 \\ u_3^0 + \beta T_2 \end{pmatrix}, \tag{3.6.53}$$

which can be solved for the temperature distribution at the next time step. The computation is then repeated for subsequent time steps; that is, for the n^{th} time step

$$\begin{pmatrix} 1 + 2\beta & -\beta & \\ -\beta & 1 + 2\beta & -\beta \\ & -\beta & 1 + 2\beta \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ u_2^{n+1} \\ u_3^{n+1} \end{pmatrix} = \begin{pmatrix} u_1^n + \beta T_1 \\ 0 \\ u_3^n + \beta T_2 \end{pmatrix}. \tag{3.6.54}$$

All this is great, but this method seems like a lot of trouble. Why do we need this much more complicated method? Well, it turns out that *implicit methods are always stable*!

3.6.5 Numerical Integration

Recall that the definite integral

$$\int_a^b f(x)\,\mathrm{d}x \tag{3.6.55}$$

is simply the area between the curve $f(x)$ and the x-axis. If we have a discretized version of $f(x)$, we can define ways to evaluate Eq. (3.6.55) geometrically.

Trapezoidal Rule

Perhaps the simplest approximation of the area is to treat the region bounded by $f(x)$, $y = 0$, $x = a$, and $x = b$ is to draw a straight line between $f(a)$ and $f(b)$ to form a trapezoid. The area of this trapezoid is then

$$\text{Area} = \left[\frac{f(b) + f(a)}{2}\right](b - a)\ . \tag{3.6.56}$$

If we divide up the region of integration between the endpoints into N equal segments, then we can compute the are for each and sum the results to approximate the integral. That is, define

$$\Delta x \equiv \frac{b - a}{N} \tag{3.6.57}$$

$$x_n \equiv a + n\Delta x \tag{3.6.58}$$

so that the integral can be approximated as

$$\int_a^b f(x)\,\mathrm{d}x \simeq \sum_{n=0}^{N-1} \frac{f(x_{n+1}) + f(x_n)}{2}\Delta x\,. \tag{3.6.59}$$

Simpson's Rule

Simpson's rule is an extension of the trapezoidal rule—essentially using a quadratic interpolation of the function on the interval rather than a linear

interpolation. A brief derivation of this result is given in Sec. B.1, but the main result is that Eq. (3.6.55) can be approximated as

$$\int_a^b f(x)\,\mathrm{d}x \simeq \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{3.6.60}$$

And as before, we can improve the approximation by breaking the region $x \in [a, b]$ into subintervals and [with the definitions Eq. (3.6.57) and Eq. (3.6.58)] we have

$$\int_a^b f(x)\,\mathrm{d}x \simeq \sum_{n=0}^{N-2} \frac{1}{6}\left[f(n_n) + 4f\left(x_{n+1}\right) + f(x_{n+2})\right]\Delta x\,. \tag{3.6.61}$$

Note that the quadratic interpolation requires knowledge of three points (rather than 2). Further, the points ***must*** be equally spaced.[†] A similar result is obtained if a third-order polynomial is fit, now requiring four points

$$\int_a^b f(x)\,\mathrm{d}x \simeq \frac{b-a}{8}\left[f(a) + 2f\left(\frac{a+b}{3}\right) + 2f\left(\frac{2(a+b)}{3}\right) + f(b)\right]. \tag{3.6.62}$$

Gaussian Quadrature

This method is a bit obscure, but it has appeared on at least two exams, once without any formulation given. The general idea is to approximate the integral as

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \simeq \sum_{i=1}^{n} w_i\, f(x_i), \tag{3.6.63}$$

where w_i are weights for each point in the interval. For Gauss-Legendre quadrature,[‡] the points x_i are the i^{th} roots of the n^{th} Legendre polynomial. The Legendre polynomials are defined by

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\left(x^2-1\right)^n\right] \tag{3.6.64}$$

and the weights are given by

$$w_i = \frac{2}{\left(1-x_i^2\right)\left[P_n'(x_i)\right]^2}\,. \tag{3.6.65}$$

The first few Legendre polynomials are listed in Table 3.8.

[†] Equation (3.6.57) implies that the points are equally spaced, but this is not a requirement for the trapezoidal rule; we could sample nonuniformly if we wanted to. However, the Lagrange interpolation (see Sec. B.1) assumes the points are equally spaced.

[‡] This is the only one they've asked about. It's probably not worth trying to memorize the other schemes for this test; frankly even this basic scheme seems like a pedantic exercise in memorization. But I don't write the test, so it's best to be prepared.

n	$P_n(x)$	n	$P_n(x)$
1	1	3	$\frac{1}{2}(3x^2-1)$
2	x	4	$\frac{1}{2}(5x^3-3x)$

Table 3.8: The first four Legendre poynomials.

Since both instances asked about 3-point quadrature, it's probably better just to memorize that for $n = 3$,

$$x_i = -\sqrt{\frac{3}{5}},\ 0,\ \sqrt{\frac{3}{5}}, \quad \text{and} \quad w_i = \frac{5}{9},\ \frac{8}{9},\ \frac{5}{9}\,. \tag{3.6.66}$$

If we need to change the interval we're operating over, this can be accomplished with a change of variables. Suppose we want to evaluate Eq. (3.6.63) over the interval $[a, b]$. Well, then let

$$y = \frac{b-a}{2}x + \frac{a+b}{2}, \tag{3.6.67}$$

so that $\mathrm{d}y = \mathrm{d}x\,(b-a)/2$, and we can compute

$$\int_a^b f(x)\,\mathrm{d}x = \frac{b-a}{2}\int_{-1}^{1} f(y)\,\mathrm{d}y\,. \tag{3.6.68}$$

Associated Errors

The error is proportional to the derivative one degree higher than the approximating polynomial. For instance, for the trapezoidal rule fits a first degree polynomial (i.e., a line) to the curve, and thus the error is proportional to the second derivative of the function. This means, these schemes are *exact* for polynomials of one higher degree (e.g., Simpon's rule has no error—regardless of the step size—when approximating a cubic function). The precise error bounds associated with these methods are given in Table 3.9.

It is not trivial to prove this formally, but the fact is related to the error incurred by truncating the Taylor series of a function at a certain power of x. That is, if we write

$$\begin{aligned} f(x) &\simeq \underbrace{f(0) + f'(0)\,x + \frac{1}{2}\,f''(0)\,x^2}_{g(x)} + \frac{1}{6}f'''(0)\,x^3 + \ldots \\ &\simeq g(x) + \mathcal{O}\left(x^3\right)\,. \end{aligned} \tag{3.6.69}$$

If we integrate both sides, we will have

$$\int f(x)\,\mathrm{d}x \simeq \int g(x)\,\mathrm{d}x + \mathcal{O}\left(x^3\right) . \tag{3.6.70}$$

To justify the last term staying as $\mathcal{O}(x^3)$, imagine that there is an increase to fourth order from the integration, and a reduction to third order by the accumulation of these errors over the interval. This is not a precise argument, and it doesn't tell us how to get the coefficients in Table 3.9, but this is the general idea.

Method	**Error Bound**
Riemann Sum	$\frac{(b-a)^2}{2N} f'(c)$
Trapezoidal Rule	$\frac{(b-a)^3}{12N^2} f''(c)$
Simpson's Rule	$\frac{(b-a)^5}{180N^4} f^{(\mathrm{iv})}(c)$

Table 3.9: Error bounds for numerical integration schemes. Note that $f^n(c)$ represents the bound of the indicated derivative over the interval, that is, $\|\mathrm{d}f/\mathrm{d}x\| \leq f'(c)$ for $x \in [a, b]$. Here Simpson's rule refers to Eq. (3.6.61), i.e., Simpson's 1/3 rule.

3.6.6 Integration of ODEs

Euler's Method

Consider the ODE

$$\dot{x} = f(x, t) . \tag{3.6.71}$$

Now suppose we know an initial value of $x_0 = x(t_0)$. Well, we expect that the value of x some short time later will be given by

$$x(t_0 + h) \simeq x(t_0) + h \left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=t_0} + \mathcal{O}\left(h^2\right) \tag{3.6.72}$$

$$\simeq x_0 + h \cdot f(t_0, x_0) . \tag{3.6.73}$$

We could repeat the process, where the n^{th} term is calculated from the values at the previous step

$$x_{n+1} \simeq x_n + h \cdot f(t_n, x_n) . \tag{3.6.74}$$

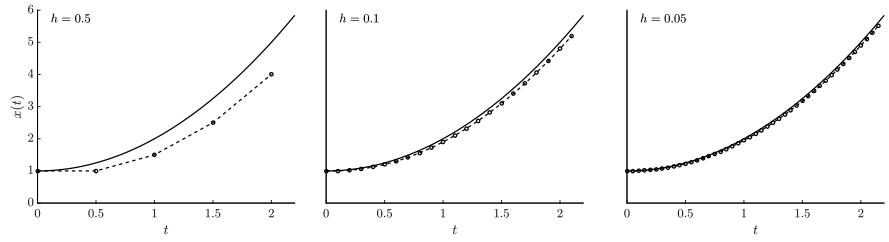


Figure 3.8: Integration of the function $x'(t) = 2t$ with initial condition $x_0 = 1$, computed with Eq. (3.6.76) and the indicated step size. The solid line represents the exact solution $x = t^2 + 1$. The error is significant for $h = 0.5$, but diminishes as smaller steps are used.

In this way we can approximate the solution to the ODE.

The Midpoint Method While Euler's method works where h is small,[†] we suspect that assuming the slope at t_n is a good approximation in the interval (t_n, t_{n+1}) might not be optimal. It would make more sense to use the slope at the midpoint of the interval, that is, at $t = t_n + h/2$. But in order to know the slope at the midpoint, we need to know the value of x at the midpoint. We first compute the value of x at midpoint:

$$x_{n+1/2} \simeq x_n + \frac{h}{2}\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=t_n} . \tag{3.6.75}$$

Then, we use $x_{n+h/2}$ to find the slope at the midpoint, and approximate the function as

$$x_{n+1} \simeq x_n + h\, f(t_{n+1/2}, x_{n+1/2}) . \tag{3.6.76}$$

Implicit Euler's Method Euler's method uses the derivative at t_n to calculate the value of x_{n+1}. But what if we used the value of the derivative at the end of the times step? That is, what about

$$x_{n+1} = x_n + f'(t_{n+1}, x_{n+1}) . \tag{3.6.77}$$

Well this works too, except we don't know what x_{n+1} is—after all, that's what we're trying to calculate in the first place. For this reason, this method is called the *implicit (or backward-Euler) technique.*

[†]That is, small enough such that the slope does not change much from t to $t + h$, i.e., $\ddot{x}/h \ll 1$.

Since we know all the other parameters in Eq. (3.6.77), we could plug them in and get a nonlinear equation for x_{n+1}. Then, we can use Newton's method (or your favorite root-finding technique from Sec. 3.6.1) to find x_{n+1}, and proceed on this way. But this is a lot harder than the explicit method—why is this a better idea than just using a tiny step size? As with any implicit method, the answer is stability.

Stability Loosely, stiff equations represent systems "involving rapidly changing components together with slowly changing ones."[18] It is usually in these cases that we need to worry about the convergence of our solution method. Following Ref. 18, we'll consider the ODE

$$x' = f(x) = -ax\,, \qquad x(0) = x_0\,, \tag{3.6.78}$$

which has the exact solution

$$x = x_0 e^{-at}\,. \tag{3.6.79}$$

The slope is much steeper near $t = 0$ than for $t \gg 0$, so we need to be ensure that our step size is appropriate. If it's too large, then the explicit Euler's method will overshoot the solution and be very wrong. So how big a step can we take?

We expect our solution to be bounded, so if we look at the ratio between subsequent function values, we have for this particular problem with the explicit Euler's method

$$\begin{aligned} x_{n+1} &= x_n + f(x_n)\,h = x_n(1 - ah) \\ &\implies x_{n+1} = (1 - ah)^n x_0\,. \end{aligned} \tag{3.6.80}$$

Since we expect our solution to decay exponentially, we require

$$h < \frac{2}{a}\,. \tag{3.6.81}$$

Now using the implicit Euler's method, we have

$$\begin{aligned} x_{n+1} &= x_n + f(x_{n+1})\,h = x_n + a x_{n+1} \\ &\implies x_{n+1} = \left(\frac{1}{1 + ah}\right)^n x_0\,. \end{aligned} \tag{3.6.82}$$

We see that this solution will dcay to 0 regardless of our step size h. This is usually the motivation to use an implicit method.

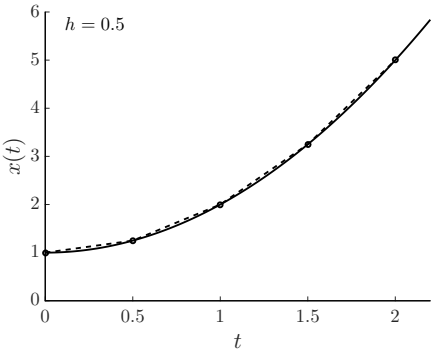


Figure 3.9: Integration of the function $x'(t) = 2t$ with initial condition $x_0 = 1$ using the Runge-Kutta method Eq. (3.6.87) with step size $h = 0.5$. Compare with the results for the same step size in Fig. 3.8.

Runge-Kutta Method

We saw that Euler's method had the basic weakness that it assumed that the slope (i.e., the derivative) of x at the starting point was a good value to use for the entire time step. But it might not be, and the midpoint method uses the idea that the slope at the midpoint of the interval is a better guess. The Runge-Kutta[†] method is essentially a weighted sum to compute the next step. It defines

$$c_1 = f\left(t_n,\, x_n\right) \tag{3.6.83}$$

$$c_2 = f\left(t_{n+h/2},\, x_n + \frac{h}{2}c_1\right) \tag{3.6.84}$$

$$c_3 = f\left(t_{n+h/2},\, x_n + \frac{h}{2}c_2\right) \tag{3.6.85}$$

$$c_4 = f\left(t_{n+h},\, x_n + hc_3\right), \tag{3.6.86}$$

and then

$$x_{n+1} \simeq x_n + \frac{h}{6}c_1 + \frac{h}{3}c_2 + \frac{h}{3}c_3 + \frac{h}{6}c_4\,. \tag{3.6.87}$$

Strictly

[†]If we're being pedantic, "Runge-Kutta" refers to the general integration scheme that's used for Euler's method and the trapezoidal rule, i.e., estimating an effective slope and computing the next point. The scheme discussed in this section is the fourth-order implicit Runge-Kutta method, but for the purposes of this exam, it is "the Runge-Kutta method".

Trapezoidal Rule

The trapedzoidal rule for ODEs is in essence the same as the numerical integration method with the same name. Rather than using the slope at the initial point as in Euler's method, we use the average of the slopes at the n^{th} and $(n+1)^{\text{th}}$ point. That is,

$$x_{n+1} = x_n + \frac{h}{2}\Big[f(x_n, t_n) + f(x_{n+1}, t_{n+1})\Big]\,. \tag{3.6.88}$$

But wait, we don't know the value of $f(x_{n+1},\, t_{n+1})$, since x_{n+1} is what we were trying to find in the first place! The trapezoidal rule is an implicit method, meaning we will have x_{n+1} on both sides of the equation. Solution of this equation requires something like Newton's method.

Predictor-Corrector Method

The predictor-corrector method is really just the trapezoidal method, but using Euler's method to find the function value at the ending time-step, rather than resorting to a nonlinear solver. That is, first the ending function value is "predicted" by

$$\bar{x}_{n+1} \simeq x_n + h \cdot f(x_n)\,. \tag{3.6.89}$$

Then the slope is evaluated for this function value and used in Eq. (3.6.88) to "correct" the value at x_{n+1}

$$x_{n+1} = x_n + \frac{h}{2}\Big[f(x_n, t_n) + f(\bar{x}_{n+1}, t_{n+1})\Big]\,. \tag{3.6.90}$$

3.6.7 Solving $A\boldsymbol{x} = \boldsymbol{b}$

Of course we can find a solution by computing $\boldsymbol{x} = A^{-1}\boldsymbol{b}$,[†] provided A is invertible and $\boldsymbol{b}$ is in the column space of A. But this can be computationally expensive; inverting a matrix by Gaussian elimination is laborious. Sometimes it's better to use iterative methods.

[†]Or if you're using MATLAB like you would be in any practical situation, typing `A\x`.

Gauss-Seidel

Note that the matrix A can be written as the sum of an upper- and lower-diagonal matrix

$$\begin{aligned} A &= L + U \\ &= \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \end{aligned} \tag{3.6.91}$$

Thus we can rearrange the matrix equation to read

$$\begin{aligned} L\boldsymbol{x} &= \boldsymbol{b} - U\boldsymbol{x} \\ &\implies \boldsymbol{x} = L^{-1}\left(\boldsymbol{b} - U\boldsymbol{x}\right). \end{aligned} \tag{3.6.92}$$

Once an initial guess has been made for $\boldsymbol{x}$, successive iterations can be made (much like Newton's method) such that

$$\boldsymbol{x}_{n+1} = L^{-1}\left(\boldsymbol{b} - U\boldsymbol{x}_n\right) \tag{3.6.93}$$

Because U and L are triangular, backward and forward substitution may be used, repsectively. However the method cannot be parallelized as the Jacobi method can.

Jacobi Method

The Jacobi method is very similar to the Gauss-Seidel method, except that the matrix A is broken into its diagonal and off-diagonal parts

$$\begin{aligned} A &= D + R \\ &= \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} + \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ a_{21} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & 0 \end{pmatrix}. \end{aligned} \tag{3.6.94}$$

Rearranging gives

$$\begin{aligned} D\boldsymbol{x} &= \boldsymbol{b} - R\boldsymbol{x} \\ &\implies \boldsymbol{x} = D^{-1}\left(\boldsymbol{b} - U\boldsymbol{x}\right). \end{aligned} \tag{3.6.95}$$

But D^{-1} is easy to compute, since D is diagonal: we just take the reciprocal of its non-zero elements. The iteration method is the same

$$\boldsymbol{x}_{n+1} = D^{-1}\left(\boldsymbol{b} - R\boldsymbol{x}_n\right) . \tag{3.6.96}$$

While this method would require us to retain $\boldsymbol{x}_n$ while we compute $\boldsymbol{x}_{n+1}$ (whereas we could overwrite it step-by-step in the Gauss-Seidel method), we can parallelize this method.

Appendices

Appendix A

Acoustics Notes

A.1 Another Derivation of the Momentum Equation

The momentum equation is essentially a statement of Newton's second law. Consider a small element of the fluid $\mathrm{d}V$ in Cartesian coordinates (i.e., a cubic test volume). The net force on its faces perpendicular to the x-axis due to the surrounding pressure are

$$\begin{aligned} \mathrm{d}F_x &= [P(x) - P(x + \mathrm{d}x)]\ \mathrm{d}y\,\mathrm{d}z \\ &= -\frac{\partial P}{\partial x}\,\mathrm{d}V\,. \end{aligned} \tag{A.1.1}$$

The procedure for y and z is identical. We could also have viscous forces, or body forces due to gravitation (or magnetic, or electric) fields, but we'll assume that these are negligible. Then we see that infinitesimal force is

$$\begin{aligned} \mathrm{d}\boldsymbol{F} &= \left(-\frac{\partial P}{\partial x}\boldsymbol{e}_x - \frac{\partial P}{\partial y}\boldsymbol{e}_y + \frac{\partial P}{\partial z}\boldsymbol{e}_z\right)\,\mathrm{d}V \\ &= -\nabla P\,\mathrm{d}V\,. \end{aligned} \tag{A.1.2}$$

Now since the force is equal to the "mass times acceleartion", we need to determine the acceleration of the fluid. If the fluid is travelling with velocity $\boldsymbol{u}(\boldsymbol{r}, t)$, then we must account for the fact that the fluid moves during the time interval $\mathrm{d}t$:

$$\begin{aligned} \boldsymbol{u}\,(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z, t + \mathrm{d}t) = \\ \boldsymbol{u}(x, y, z, t) + \frac{\partial \boldsymbol{u}}{\partial x}\frac{\partial x}{\partial t}\,\mathrm{d}t + \frac{\partial \boldsymbol{u}}{\partial y}\frac{\partial y}{\partial t}\,\mathrm{d}t + \frac{\partial \boldsymbol{u}}{\partial z}\frac{\partial z}{\partial t}\,\mathrm{d}t + \frac{\partial \boldsymbol{u}}{\partial t}\,\mathrm{d}t\,. \end{aligned} \tag{A.1.3}$$

But the terms $\partial x/\partial t$, $\partial y/\partial t$, and $\partial z/\partial t$ are the components of the velocity. So if we rearrange and divide Eq. (A.1.3) by $\mathrm{d}t$,[†] we can write it as

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} &= \frac{\partial \boldsymbol{u}}{\partial t} + u_x \frac{\partial \boldsymbol{u}}{\partial x} + u_y \frac{\partial \boldsymbol{u}}{\partial y} + u_z \frac{\partial \boldsymbol{u}}{\partial z} \\ &= \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u}. \end{aligned} \tag{A.1.4}$$

This is often called the "material derivative" of $\boldsymbol{u}$, written $\mathrm{D}\boldsymbol{u}/\mathrm{D}t$, i.e.,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \equiv \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\, \boldsymbol{u}\,. \tag{A.1.5}$$

Really this is just a total derivative that includes motion of the fluid (or reference frame). Unless we have any additional flow to consider (i.e., we moving with the fluid, or the fluid has some net motion), the material derivative reduces to a partial time derivative.

A.2 Adiabatic Compression

Following Schroder,[19] we first note that the first law of thermodynamics states that

$$\Delta U = Q + W = W\,, \tag{A.2.1}$$

where ΔU is the change in internal energy of the gas, Q is the heat flow ($= 0$ since the process is adaibatic), and W is the work done by the gas. Now by the equipartition theorem states that

$$U = \frac{f}{2} N k_B T\,, \tag{A.2.2}$$

where f is the number of degrees of freedom per molecule,[‡] and k_B is Boltzman's constant. Thus the infinitesimal change in energy as the gas goes from T to $T + \mathrm{d}T$ is

$$\mathrm{d}U = \frac{f}{2} N k_B\, \mathrm{d}T\,. \tag{A.2.3}$$

[†] Please send your complaints about the wanton recklessness of this maneuver to:
American Mathematical Society, c/o Complaints Dept.
201 Charles Street
Providence, RI USA

[‡]For a monotomic gas like hydrogen, $f = 3$, while for a diatomic gas like oxygen, $f = 5$.

But by Eq. (A.2.1), this the increment of work done on the gas. So then

$$\begin{aligned} \mathrm{d}U = \frac{f}{2} N k_B \,\mathrm{d}T &= -P\,\mathrm{d}V \\ \implies \frac{f}{2} N k_B \,\mathrm{d}T &= -\left(\frac{N k_B T}{V}\right) \mathrm{d}V \\ \implies \frac{f}{2}\frac{1}{T}\,\mathrm{d}T &= -\frac{1}{V}\,\mathrm{d}V\,. \end{aligned} \tag{A.2.4}$$

Integrating Eq. (A.2.4) from some reference state, we have

$$\begin{aligned} \frac{f}{2}\int_{T_0}^{T} \frac{\mathrm{d}T'}{T'} &= -\int_{V_0}^{V} \frac{\mathrm{d}V'}{V'} \\ V T^{f/2} &= V_0 T_0^{f/2}\,. \end{aligned} \tag{A.2.5}$$

Finally from the ideal gas law,

$$T = \frac{PV}{N k_B} \tag{A.2.6}$$

so that Eq. (A.2.5) gives

$$\begin{aligned} V\left(\frac{PV}{N k_B}\right)^{f/2} &= V_0\left(\frac{P_0 V_0}{N k_B}\right)^{f/2} \\ V^{1+f/2} P^{f/2} &= V_0^{1+f/2} P_0^{f/2} \\ \implies PV^{\gamma} &= \text{const.}, \end{aligned} \tag{A.2.7}$$

where $\gamma = (f+2)/f$.

A.3 Spherical Wave Intensity

The claim was made in Sec. 2.2.5 that the expression for the time averaged intensity [Eq. (2.2.105), repeated here]

$$I_{\text{avg}} = \frac{p_{\text{rms}}^2}{\rho_0 c_0} \tag{A.3.1}$$

is true for planar and spherical waves of arbitrary form (i.e., they need not be harmonic). We will show this To verify the claim[†] for spherical waves, where $p \neq \rho_0 c_0 u$, note that the velocity potential for an outgoing spherical wave is

$$\phi = \frac{f(t - r/c_0)}{r}, \tag{A.3.2}$$

[†]Derivation here follows Ref. 4; see there for a more thorough discussion.

and therefore the pressure is

$$p = -\rho_0 \frac{\partial \phi}{\partial t} = -\rho_0 \frac{f'(t - r/c_0)}{r}, \tag{A.3.3}$$

and the (radial) particle velocity is

$$u = \frac{\partial \phi}{\partial r} = -\frac{f(t - r/c_0)}{r^2} - \frac{f'(t - r/c_0)}{r c_0}. \tag{A.3.4}$$

As before, the time-averaged intensity is, from the definition [Eq. (2.2.104)]

$$I_{\text{avg}} = \frac{1}{T} \int_0^T p\, u \, \mathrm{d}t. \tag{A.3.5}$$

The integrand becomes, from Eqs. (A.3.3) and (A.3.4),

$$\begin{aligned} pu &= \left[-\rho_0 \frac{f'(t - r/c_0)}{r} \right] \cdot \left[-\frac{f(t - r/c_0)}{r^2} - \frac{f'(t - r/c_0)}{r c_0} \right] \\ &= \frac{\rho_0}{r} \underbrace{\frac{f'(t - r/c_0)}{r}}_{\phi_t} \underbrace{\frac{f(t - r/c_0)}{r}}_{\phi} + \frac{\rho_0}{c_0} \left[\frac{f'(t - r/c_0)}{r} \right]^2 \\ &= \frac{\rho_0}{r} \phi_t \phi + \frac{1}{\rho_0 c_0} \underbrace{\left[\rho_0 \frac{f'(t - r/c_0)}{r} \right]^2}_{p^2} \\ &= \frac{\rho_0}{r} \phi_t \phi + \frac{p^2}{\rho_0 c_0}. \end{aligned} \tag{A.3.6}$$

If we substitute Eq. (A.3.6) into Eq. (A.3.5), we see that the second term will look familiar

$$\frac{1}{T} \int_0^T \frac{p^2}{\rho_0 c_0} \, \mathrm{d}t = \frac{p_{\text{rms}}^2}{\rho_0 c_0}. \tag{A.3.7}$$

If our claim that Eq. (A.3.1) holds for progressive waves, the the first term must vanish when integrated. That is

$$\frac{1}{T} \int_0^T \left(\frac{\rho_0}{r} \phi_t \phi \right) \mathrm{d}t = \frac{\rho_0}{r} \frac{\frac{1}{2}\phi^2(T) - \frac{1}{2}\phi^2(0)}{T} \stackrel{?}{=} 0. \tag{A.3.8}$$

Clearly, whether or not Eq. (A.3.8) vanishes depends on our choice of T. In the case of a harmonic signal, we choose T to be the period, and the term will indeed be 0. If the signal is not periodic, but if we allow T to become very large, then the integral will again vanish, and we will be left only with second term, and Eq. (A.3.1) will remain a true statement.

Acoustics Crib Sheet

Linear Acoustics

Need three equations for lossless wave equation:

1. Continuity $\quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0 \quad (1)$
2. Momentum $\quad \rho \frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + \nabla P = \mathbf{f} \quad (2)$
3. Equation of State $\quad P = P(\rho, S) \quad (3)$

Derive linear wave equation by letting $P = P_0 + p$, $\boldsymbol{v} = \boldsymbol{u}$, $\rho = \rho_0 + \rho'$, and retaining only first term of adiabatic (S constant) expansion of Eq. (3), so that $p = Ks$. Then discard products of acoustic quantities and using Eq. (1) to eliminate terms in Eq. (2). Solutions for planar and spherical harmonic ($\propto e^{-i\omega t}$, $k = \omega/c$) waves are

$$p(\boldsymbol{r}) = A\, e^{\pm i \boldsymbol{k}\cdot\boldsymbol{r}} \qquad \text{and} \qquad p(r) = A\frac{e^{\pm ikr}}{r}$$

The acoustic **impedance** $Z \equiv p/u$, where u is found from linearized momentum equation $\rho_0 \boldsymbol{u}_t = \nabla p$

$$Z = \pm \rho_0 c_0 \qquad Z = \rho_0 c_0 \left(1 - 1/ikr\right)^{-1} .$$

The small signal **speed of sound** is defined at adiabatic conditions $c^2 \equiv (\partial P/\partial \rho)_s = K/\rho$. For a perfect gas then $c_0 = \sqrt{\gamma RT} = \sqrt{\gamma P_0/\rho_0}$. More generally, we have PV^γ constant, and $[B/2A = (\gamma-1)/2]$

$$\begin{aligned} p &= P_0 \left[\gamma s + \frac{1}{2}\gamma(\gamma-1)s^2 + \ldots\right] \\ &\simeq \gamma P_0\, s + \frac{\gamma P_0}{2}(\gamma-1)s^2 = K\, s + \frac{B}{2A}\, K\, s^2 . \end{aligned}$$

Acoustic Quanities

RMS quantities defined by $p_{\mathrm{rms}}^2 = (1/T)\int_0^T p^2(t)\,\mathrm{d}t$. SPL defined by $L_p \equiv 10\log_{10} p_{\mathrm{rms}}^2/p_{\mathrm{ref}}^2$. For incoherent sources, $p_{\mathrm{rms}}^2 = \sum_n p_{n,\mathrm{rms}}^2$. If phase difference ϕ between 2 sources, $p_{\mathrm{rms}}^2 = p_{1,\mathrm{rms}}^2 + p_{2,\mathrm{rms}}^2 + P_1 P_2 \cos\phi$. For harmonic sources, $p_{\mathrm{rms}} = P_0/\sqrt{2}$.

1. Energy Density [J/m^3] $\quad \mathcal{E} = \frac{1}{2}\rho_0\left[u^2 + (p/\rho_0 c_0)^2\right]$
2. Intensity [W/m^3] $\quad \boldsymbol{I} = p\boldsymbol{u}$
   (a) Progressive waves: $\quad \langle I\rangle = P^2/2\rho_0 c_0$
   (b) Harmonic Waves: $\quad \langle I\rangle = \frac{1}{2}\,\mathrm{Re}\, PU^*$
3. Power [W] $\quad W = \int_S \boldsymbol{I}\cdot\boldsymbol{n}\,\mathrm{d}S$
4. Directivity [Unitless] $\quad D = D(\theta,\phi)$

Reflection and Transmission

In all cases, we need continuity of normal particle velocity. Usually we also have continuity of pressure (or there is a pressure differential). Acoustic impedance for plane waves is $Z_{\mathrm{ac}} \equiv \rho_0 c_0/\cos\theta = \rho_0 c_0/S$, and it turns out that the reflection coefficient is

$$\mathcal{R} = (Z_{\mathrm{ac},2} - Z_{\mathrm{ac},1})/(Z_{\mathrm{ac},2} + Z_{\mathrm{ac},1}) .$$

If the second medium is harder, we have a **critical angle**

$$\sin\theta_c = c_1/c_2 , \qquad \cos\theta_t = \sqrt{1 - (c_2/c_1)\sin^2\theta_i} .$$

Power reflection and transmission coefficients are governed by

$$r = \|\mathcal{R}\|^2 \qquad \tau + r = 1 .$$

The **angle of intromission** is when $r = 0$

$$\sin\theta_p = \left[1 - (Z_1/Z_2)^2\right] \Big/ \left[1 - (\rho_1/\rho_2)^2\right] .$$

For a locally reacting surface $Z_{\mathrm{ac},2} = Z_n = r_n - ix_n$ so that

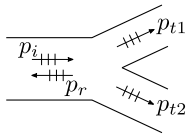
$$\mathcal{R} = (Z_n - Z_{\mathrm{ac},2})/(Z_n + Z_{\mathrm{ac},2}) .$$

The impedance seen looking into a termination is

$$Z_{\mathrm{in}} = iZ_0 \cot kL \text{ (rigid)} \qquad Z_{\mathrm{in}} = -iZ_0 \tan kL \text{ (free)}$$

If we have a pressure discontinuity, then write a force balance

$$\begin{aligned} p_i + p_r - p_t &= (m/S)\ddot{\xi} = (m/S)\dot{u} && \text{(mass law)} \\ &= r_f u && \text{(flow resistance)} \\ &= m\dot{u}\left[1 - (v\sin\theta_i/c_0)^4\right] && \text{(stiff plate).} \end{aligned}$$



If there is a change in area, need continuity of pressure and total volume velocity:

$$\begin{aligned} p_i + p_r &= p_{t1} = p_{t2} \\ S_1(u_i + u_r) &= S_1 u_{t1} + S_2 u_{t2} \end{aligned}$$

In this case we'd find

$$\mathcal{R} = (Z_{\mathrm{ac},0}^{-1} - Z_{\mathrm{ac},1}^{-1} - Z_{\mathrm{ac},2}^{-1})/(Z_{\mathrm{ac},0}^{-1} + Z_{\mathrm{ac},1}^{-1} + Z_{\mathrm{ac},2}^{-1}) .$$

Waveguides

For a rectangular (z infinite) waveguide,

$$p = \sum_{n,m} C_{mn} \left\{ \begin{array}{c} \cos k_x x \\ \sin k_x x \end{array} \right\} \left\{ \begin{array}{c} \cos k_y y \\ \sin k_y y \end{array} \right\} e^{i\sqrt{\omega^2/c_0^2 - k_x^2 - k_y^2}\, z}$$

and for a rigid circular duct with radius a ($k_r = \alpha'_{mn}/a$)

$$p = \sum_{n,m} C_{mn} J_n(k_r r) \left\{ \begin{array}{c} \cos m\theta \\ \sin m\theta \end{array} \right\} e^{i\sqrt{\omega^2/c_0^2 - k_r^2}\, z}$$

In both cases the phase speed is $c_p = \omega/k_z$ and the speed at which energy propagates is $\mathrm{d}\omega/\mathrm{d}k_z$. The **cutoff frequency** for a mode is ω such that the propagating wavenumber is real.

Radiation

The Rayleigh Integral is ($\mathscr{r} = \|\boldsymbol{r} - \boldsymbol{r}'\|$)

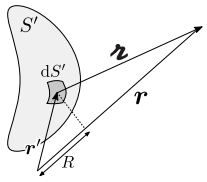
$$p(\boldsymbol{r}) = -\frac{i\omega\rho_0}{4\pi}\int_{S'} \frac{e^{ik\mathscr{r}}}{\mathscr{r}} \boldsymbol{u}(\boldsymbol{r}')\cdot\boldsymbol{n}\,\mathrm{d}S' . \quad (4)$$

Now define $\boldsymbol{e}_r = \boldsymbol{r}/\|\boldsymbol{r}\|$, and $R = \boldsymbol{r}'\cdot\boldsymbol{e}_r$, then for phase

$$\begin{aligned} \mathscr{r} &= \left[(\boldsymbol{r}-\boldsymbol{r}')\cdot(\boldsymbol{r}-\boldsymbol{r}')\right]^{\frac{1}{2}} = r\left[1 - (r'/r)^2 + 2\left(\boldsymbol{r}\cdot\boldsymbol{r}'\right)/r^2\right]^{\frac{1}{2}} \\ &\simeq r\left[1 - R/r + \mathcal{O}\left(r'^2/r^2\right)\right] = r - R , \end{aligned} \quad (5)$$

and $\mathscr{r} \simeq r$ for the denominator. Eq. (5) $\to$ Eq. (4) gives

$$\begin{aligned} p(\boldsymbol{r}) &\simeq -i\omega\rho_0 \frac{e^{ikr}}{4\pi r}\int_{S'} e^{-ikR}\boldsymbol{u}(\boldsymbol{r}')\cdot\boldsymbol{n}\,\mathrm{d}S' \\ &\simeq -i\omega\rho_0 \frac{e^{ikr}}{4\pi r}\int_{S'} \left[1 + ikR + \frac{1}{2}(ikR)^2\right]\boldsymbol{u}(\boldsymbol{r}')\cdot\boldsymbol{n}\,\mathrm{d}S' . \end{aligned} \quad (6)$$



The first, second, and third terms of Eq. (6) are the monopole, dipole and quadrupole terms. Note that for a monopole source, the dipole term cancels since there is a $-R$ point for every R. For a dipole, the monopole term vanishes, since there is no net fluid flow through S'.

For a **circular piston** of radius a, the on-axis and far-field evaluation of Eq. (4) give (respectively)

$$p(z,t) = \rho_0 c_0 u_0 \left(e^{ikz} - e^{ik\sqrt{z^2+a^2}}\right) e^{-i\omega t} ,$$

$$p(r,\theta,t) = -\frac{i\rho_0 c_0 u_0 R}{\pi r \sin\theta}\, J_1(ka\sin\theta)\, e^{-i(\omega t - kr)} .$$

The radiation impedance seen by the piston is

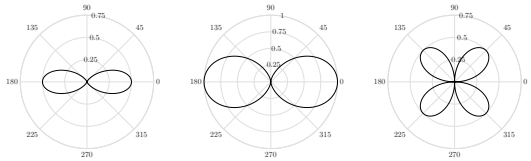
$$Z_{\mathrm{rad}} = \rho_0 c_0 \left[R_1(2ka) - iX_1(2ka)\right] .$$

In the far field, acoustic monopole and dipole field amplitudes are,

$$|p(r,\theta)| = A \quad \text{and} \quad |p(r,\theta)| = Akd\cos\theta$$

where $A = -i\omega\rho_0 c_0/4\pi r$. The lateral and longitudinal quadrupole fields are

$$|p(r,\theta)| = A(kd)^2\cos\theta\sin\theta \quad \text{and} \quad |p(r,\theta)| = A(kd)^2\cos^2\theta .$$



Appendix B

Math Notes

B.1 Derivation of Simpson's Rule

The idea of Simpson's rule is that it's relatively easy to find the area under a quadratic function, and a quadratic interpolation is a better approximation of the function over the interval Δx than is a linear interpolation. Suppose we want to interpolate over $x \in [x_0, x_2]$, with the midpoint at x_1; that is

$$x_1 = \frac{x_0 + x_2}{2} \tag{B.1.1}$$

We'll need the function values $f(x_0)$, $f(x_1)$, and $f(x_2)$ and require that our interpolation curve $g(x)$ pass through those points [i.e., $f(x_0) = g(x_0)$, etc.]. The interpolated function can be computed with Lagrange polynomials,[†] which fit N data points by a polynomial $L(x)$ defined by

$$L(x) = \sum_{n=0}^{N-1} f(x_n)\ell_n(x), \tag{B.1.2}$$

where the basis polynomials $\ell_n(x)$ are defined by

$$\ell_n(x) \equiv \prod_{\substack{m=0 \\ m\neq n}}^{N-1} \frac{x - x_m}{x_n - x_m}. \tag{B.1.3}$$

[†]A derivation of this theory is not presented here; I gotta stop somewhere or we'd be going back to Peano axioms every other line. For more on Lagrange polynomials, see, e.g., Ref. 16, Section 14.5.

Since we have three points, we can write explicitly

$$\ell_0(x) = \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right) \tag{B.1.4}$$

$$\ell_1(x) = \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right) \tag{B.1.5}$$

$$\ell_2(x) = \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right). \tag{B.1.6}$$

Then from Eq. (B.1.3),

$$\begin{aligned} g(x) = f(x_0)&\left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right) \\ &+ f(x_1)\left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right) + f(x_3)\left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right). \end{aligned} \tag{B.1.7}$$

Equation (B.1.7) can be integrated (with some effort) to give

$$\int_a^b g(x)\,\mathrm{d}x = \frac{b-a}{6}\left[f(x_0) + 4f(x_1) + f(x_2)\right]. \tag{B.1.8}$$

This result forms the argument of the summation of Eq. (3.6.61).
The error incurred by computing a definite integral by approximating the integrand $f(x)$ with a polynomial $P(x)$ on the interval $x = \{x_0, x_1, \ldots, x_N\}$ is

$$\begin{aligned} E &= \int_a^b \left[f(x) - P(x)\right]\,\mathrm{d}x \\ &= \int_a^b \frac{f^{(n+1)}(c)}{(n+1)!}\prod_{n=0}^{n}(x - x_n)\,\mathrm{d}x, \end{aligned} \tag{B.1.9}$$

where $c = (a + b)/2$ is the midpoint.

B.2 Derivation of Taylor's Theorem

Derivation is due to Bob Prego, posted on Mathematics StackExchange on 13 September 2013.

The fundamental theorem of calculus states that

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a), \tag{B.2.1}$$

where $f' = \mathrm{d}f/\mathrm{d}x$. Rearranging,

$$f(b) = f(a) + \int_a^b f'(x_1)\,\mathrm{d}x_1\,, \tag{B.2.2}$$

where x_1 is just the dummy variable of integration. By another application of the fundamental theorem of calculus,

$$f'(x_1) = f'(a) + \int_a^{x_1} f''(x_2)\,\mathrm{d}x_2\,, \tag{B.2.3}$$

so that Eq. (B.2.2) becomes

$$\begin{aligned} f(b) &= f(a) + \int_a^b \left(f'(a) + \int_a^{x_1} f''(x_2)\,\mathrm{d}x_2\right)\mathrm{d}x_1 \\ &= f(a) + \int_a^b f'(a)\,\mathrm{d}x_1 + \int_a^b \int_a^{x_1} f''(x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1\,. \end{aligned} \tag{B.2.4}$$

We ca repeat this process to find successive expressions for $\int_a^{x_n} f^{(n)}(x_n)\,\mathrm{d}x_n$; e.g., doing so for the last term in Eq. (B.2.4) gives

$$\begin{aligned} f(b) = f(a) + \int_a^b f'(a)\,\mathrm{d}x_1 + \int_a^b \int_a^{x_1} f''(a)\,\mathrm{d}x_2\,\mathrm{d}x_1 + \\ \int_a^b \int_a^{x_1} \int_a^{x_3} f'''(x_3)\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1\,. \end{aligned} \tag{B.2.5}$$

Now,

$$\begin{aligned} \int_a^b f'(a)\,\mathrm{d}x_1 &= f'(a)\int_a^b \mathrm{d}x_1 \\ &= f'(a)(b-a), \end{aligned} \tag{B.2.6}$$

and similarly

$$\begin{aligned} \int_a^b \int_a^{x_1} f''(a)\,\mathrm{d}x_2\,\mathrm{d}x_1 &= f''(a)\int_a^b \int_a^{x_1} \mathrm{d}x_2\,\mathrm{d}x_1 \\ &= f''(a)\int_a^b (x_1 - a)\,\mathrm{d}x_1 \\ &= f''(a)\left(\frac{x_1^2}{2} - a x_1\Big|_a^b\right) \\ &= f''(a)\left[\left(\frac{1}{2}b^2 - ab\right) - \left(\frac{1}{2}a^2 - a^2\right)\right] \\ &= f''(a)\,\frac{(a-b)^2}{2}\,. \end{aligned} \tag{B.2.7}$$

In fact

$$\int_a^b \int_a^{x_1} \cdots \int_a^{x_{n-1}} f^{(n)}(a)\, \mathrm{d}x_{n-1} \ldots \mathrm{d}x_2\, \mathrm{d}x_1 = f^{(n)}(a)\, \frac{(a-b)^n}{n!}\,. \tag{B.2.8}$$

We then see from Eqs. (B.2.2) to (B.2.5) and (B.2.8) that the series expansion has the familiar form

$$f(b) = f(a) + \sum_{n=1}^{\infty} f^{(n)}(a)\, \frac{(a-b)^n}{n!}\,. \tag{B.2.9}$$

If we truncate the series after a finite number of terms N, then we see that the function value is given by

$$f(b) = S_N(b) + R_N(b), \tag{B.2.10}$$

where S_N is the N^{th} partial sum [i.e., the truncation of Eq. (B.2.9) after N terms], and R_N is the remainder. The remainder is the error in the approximation, and is given by the sum of the terms we did not include in the summation. It can be shown that

$$R_N(b) = f^{(N+1)}(x)\, \frac{(b-a)^{N+1}}{(N+1)!} \tag{B.2.11}$$

for $x \in [a, b]$.

B.3 Numerical PDE Stability

A thorough discussion of stability is out of scope for this exam, but it's useful I think to know the rudiments of stability analysis, so that the requirements in Eq. (2.4.67) aren't mysterious. Loosely, ***stability*** is a measure of the local error to remain bounded as the solution at subsequent time steps is calculated.[†]

The stability referred to on previous exams seems to be ℓ^2 stability (the other common type is ℓ^∞ stability; ℓ^∞ stability implies ℓ^2 stability and is a stronger condition). My general understanding is that ℓ^2 stability is usually sufficient, but will have some trouble near discontinuous initial conditions (much as a finite Fourier series gives rise to Gibbs phenomena when approximating a function with a sharp transition).

[†] A more precise statement is beyond what I'm confident to be able to summarize. But consult a text on numerical PDEs (e.g., Refs. 20 and 21) for more accurate/complete information.

ℓ^2 Stability: von Neumann Analysis

Suppose we're solving the 1D heat equation with a first-order explicit scheme [Eq. (3.6.45)]

$$u_m^{n+1} = u_m^n + \beta\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right) \tag{B.3.1}$$

subject to some initial condition u^0. Now suppose some small perturbation of u_m^n, call it v_m^n, also satisfies the finite difference equation. Then, since the finite difference equation is linear, we have that their difference $\epsilon_m^n = u_m^n - v_m^n$ is also a solution, i.e.,

$$\epsilon_m^{n+1} = \epsilon_m^n + \beta\left(\epsilon_{m+1}^n - 2\epsilon_m^n + \epsilon_{m-1}^n\right) . \tag{B.3.2}$$

Suppose now we let $E^n(x)$ be the semi-continuous version of ϵ, i.e., $\epsilon_m^n = E^n(x_m)$. Take the spatial Fourier transform, where $\hat{f}(\xi) = \mathcal{F}[f(x)]$, of both sides of Eq. (B.3.2)

$$\begin{aligned}
\hat{E}^{n+1}(\xi) &= \beta\mathcal{F}\left[E^n(x+\Delta x)\right] + (1-2\beta)\hat{E}^n(\xi) + \beta\mathcal{F}\left[E^n(x-\Delta x)\right] \\
&= \beta e^{i\xi\Delta x}\hat{E}^n(\xi) + (1-2\beta)\hat{E}^n + \beta e^{-i\xi\Delta x}\hat{E}^n(\xi) \\
&= \Big[\beta\underbrace{\left(e^{i\xi\Delta x} + e^{-i\xi\Delta x}\right)}_{2\cos\xi\Delta x} + 1 - 2\beta\Big]\hat{E}^n \\
&= \left[1 - 2\beta(1-\cos\xi\Delta x)\right]\hat{E}^n .
\end{aligned} \tag{B.3.3}$$

The quantity in brackets is called the *Fourier sign*, and convergence is assured if this value is less than or equal to 1.[†] We see that Fourier sign will remain less than unity if $\beta \leq 1/2$, which is the condition we claimed before.

ℓ^∞ Stability: The Maximum Principle

Taking a perturbation of the finite difference scheme as above, but now with an *implicit* first order discretization of the 1D heat equation, we find

$$\epsilon_{m+1}^{n+1} = \frac{\beta}{1+2\beta}\left(\epsilon_{m+1}^{n+1} + \epsilon_{m-1}^{n+1} + \epsilon_m^n\right) . \tag{B.3.4}$$

Note that the error at a time point is a convex average of the surrounding points (in time and space). Therefore, suppose ϵ_{m+1}^{n+1} is an absolute maximum or minimum. Since this value is an average of its surrounding points, then

[†]This has to do with Parseval's theorem, which (in spirit) says that $\int |f|^2\,\mathrm{d}x = \int |\hat{f}|^2\,\mathrm{d}\xi$, i.e., that the "energy" in a function and its Fourier transform are equal.

those surrounding points must also be absolute maxima or minima. By induction we can say this for all the interior points, and therefore that if ϵ has an extremum that's not on the spatial boundary or occurs at $t = 0$, then the error must be constant. Thus the implicit method ensures that any error we start out with will either remain the same or decrease in magnitude—which ensures stability.

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"*When in doubt, draw a picture, and then integrate by parts.*"
— John McCuan